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Propagation of ultrasonic Love waves in nonhomogeneous elastic functionally graded materials

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ABSTRACT

This paper presents a theoretical study of the propagation behavior of ultrasonic Love waves in nonhomogeneous functionally graded elastic materials, which is a vital problem in the mechanics of solids. The elastic properties (shear modulus) of a semi-infinite elastic half-space vary monotonically with the depth (distance from the surface of the material). The Direct Sturm–Liouville Problem that describes the propagation of Love waves in nonhomogeneous elastic functionally graded materials is formulated and solved by using two methods: i.e., (1) Finite Difference Method, and (2) Haskell-Thompson Transfer Matrix Method.

The dispersion curves of phase and group velocity of surface Love waves in inhomogeneous elastic graded materials are evaluated. The integral formula for the group velocity of Love waves in nonhomogeneous elastic graded materials has been established. The effect of elastic non-homogeneities on the dispersion curves of Love waves is discussed. Two Love wave waveguide structures are analyzed: (1) a nonhomogeneous elastic surface layer deposited on a homogeneous elastic substrate, and (2) a semi-infinite nonhomogeneous elastic half-space. Obtained in this work, the phase and group velocity dispersion curves of Love waves propagating in the considered nonhomogeneous elastic waveguides have not previously been reported in the scientific literature. The results of this paper may give a deeper insight into the nature of Love waves propagation in elastic nonhomogeneous functionally graded materials, and can provide theoretical guidance for the design and optimization of Love wave based devices.

1. Introduction

Shear horizontal (SH) surface Love waves have long been used in many fields of science and technology, e.g., in geophysics [1], seismology [2], non-destructive testing (NDT) of materials [3,4] and for determining the physical properties of materials. Sensors based on Love waves (due to their high sensitivity) are used for measuring physical properties of the liquid (e.g., viscosity and density) [5–7] as biosensors [8] and chemo-sensors [9], to investigate of thin films [10] and layers produced in the surface region of the substrate as a result of various technological processes (diffusion, implantation, carburizing, nitriding, shot peening, the laser treatment, etc.) [11], and also for testing of composites [12]. The use of layered Love waves waveguides with a nonhomogeneous distribution of physical properties can significantly improve performance (e.g., sensitivity and selectivity) of bio and chemosensors that employ the inhomogeneous elastic waveguides [13].

SH surface acoustic waves (Love and Bleustein–Gulyaev type) may also be used to study spatial profiles changes in mechanical properties (e.g., modulus of elasticity and density) of the Functionally Graded Material (FGM) [14–17]. These materials are heterogeneous media, in which the mechanical parameters are functions of the distance from the surface into the bulk of the material. Functionally graded materials can provide elevated mechanical properties (e.g., high strength and hardness) and superior exploitation characteristics (e.g., crack, wear and corrosion resistance). The FGM are widely used in modern industry (e.g., automotive, aviation, aerospace and electronic) [18]. Love wave penetration depth depends on the frequency. Thus, by changing the frequency of the wave one can probe subsurface profiles of materials. Love wave energy is concentrated near the surface of the waveguide. For this reason, any disturbance in the material parameters in the surface region have considerable impact on the dispersion characteristics of the Love wave (i.e., velocity and attenuation). Therefore, the Love waves are particularly convenient to study the physical properties of inhomogeneous graded materials.

The aim of this study was to develop a theoretical model of the propagation of SH (Shear Horizontal) surface Love waves in

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functionally graded materials with a monotonic variation of the elastic properties with the depth (distance from the treated surface of the material).

Determination of the phase velocity dispersion curves and the distribution of the mechanical displacement (into the bulk of the material) of the Love wave, for known profiles of elastic parameters of the medium in which the Love wave propagates, constitutes a Direct Sturm–Liouville Problem. In this study, the following profiles of the elastic coefficient $c_{44}(x)$ were considered: (1) the square root profile $n = 1/2$, (2) linear profile $n = 1$, (3) quadratic profile $n = 2$, (4) power type profile $n = 10$, (5) step profile $n = \infty$, (6) exponential profile, (7), profile of the $1/\cosh^2$ type (similar to the Gaussian profile). Governing equations of motion and the appropriate boundary conditions are given. The Direct Sturm–Liouville Problem has been solved using the Finite Difference Method and Transfer Matrix Method (Haskell-Thompson) [19,20]. The article includes a comparison of Love wave dispersion curves derived by these two numerical methods. Moreover, the integral formula for the group velocity of Love waves propagating in elastic graded materials was established, which is a novelty.

The problem of the Love wave propagation in nonhomogeneous elastic graded media was previously analyzed using various approximate methods such as the method of Frobenius [21], the method of Peano [22] and the WKB method [23]. However, these methods are pre-computer era methods. Their use does not introduce significant advantages in relations with modern numerical methods such as Finite Difference Method (FDM), Finite Element Method (FEM) or the Transfer Matrix Method (TMM). TMM method was used to analyze the Love wave propagation in nonhomogeneous medium [24]. However, this study presents only a general description of the TMM method without giving specific examples of physically realistic nonhomogeneous profiles and the corresponding dispersion curves.

In this work the problem of Love wave propagation in nonhomogeneous graded media was solved using the TMM and FDM methods. Phase and group velocity dispersion curves of Love waves in selected nonhomogeneous elastic graded media (power-law type profiles, exponential profile and the profile of the type 1/cosh 2) were evaluated. According to the authors' best knowledge, evaluation of the phase and group velocity dispersion curves for these selected profiles of the elastic modulus $c_{44}(x)$ is a novelty.

The results obtained in this work can constitute the basis of the inverse procedure (Inverse Sturm–Liouville Problem) to determine profiles (as a function of depth) of the mechanical properties of

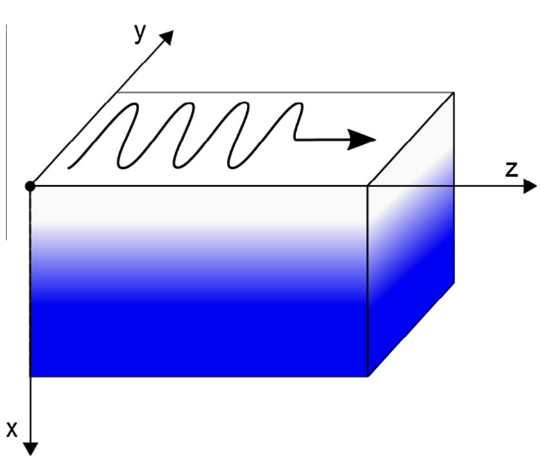

Fig. 1. Geometry of the Love wave waveguide structure (inhomogeneous elastic half-space), and coordinate system.

inhomogeneous FGM resulting from the application of various technological processes of surface treatment. The results of this study also provide a more complete description (than those published in the scientific literature) of the propagation of Love waves in graded materials with various profiles of changes in elastic properties, e.g., in layered inhomogeneous microstructures used in MEMS (Micro Electro Mechanical Systems) and other microelectronic devices, in photonics and in acoustoelectronics [3,25].

The results of this study can also find application in geophysics, earthquake engineering [26] and seismology to investigate the internal structure of Earth. Moreover, they can be very helpful in exploration of natural resources (e.g., gas and petroleum) [27].

Due to the similarity of the mathematical description of the phenomenon of propagation of Love waves in elastic inhomogeneous media and a description of the propagation of light waves in inhomogeneous planar optical waveguides, established in this work the theory of Love waves in elastic inhomogeneous media can also be used to analyze performance of inhomogeneous optical planar waveguides [28].

The results obtained in this paper are novel and fundamental and can give more profound insight into the nature of Love wave propagation in the elastic nonhomogeneous media (e.g., in functionally graded materials and composites).

Section 2 presents a mathematical model of the propagation of Love waves in the graded materials formulated as a Direct Sturm–Liouville Problem. Section 3 shows the considered shear modulus profiles $c_{44}(x)$ in the graded medium. Description of numerical methods applied to solve the Direct Sturm–Liouville Problem (i.e., the Finite Difference Method and Transfer Matrix Method) is included in Section 4. Section 5 contains the results of numerical calculations and discussion of the results. Finally, conclusions are presented in Section 6.

2. Direct Sturm–Liouville Problem

Love wave propagation in inhomogeneous elastic media can be described in terms of the Sturm–Liouville Direct Problem. Determination of the phase velocity and mechanical displacement distribution with depth of the SH surface Love wave from a knowledge of elastic parameters of a non-homogeneous half-space constitutes a Direct Sturm–Liouville Problem.

2.1. Love waves

2.1.1. Formulation of the problem

Consider the Love wave that propagates in a nonhomogeneous elastic half-space, as shown in Fig. 1. The elastic properties of inhomogeneous half-space vary monotonically with depth (distance from the surface).

Mechanical vibrations of the SH surface Love wave are performed along the y axis perpendicularly to the direction of propagation z and parallel to the propagation surface. The x axis is normal to the waveguide surface.

Mathematical description of the propagation of surface shear Love waves in graded media involves the use of continuum mechanics formalism to describe the motion of inhomogeneous elastic half-space.

SH surface wave of the Love type which propagates in a nonhomogeneous waveguide structure of Fig. 1 may be represented in the following form: $v(x, z, t) = V(x) \cdot \exp j(\beta z - \omega t)$, where: $V(x)$ is the distribution of the mechanical displacement of the Love wave with the depth, β is the wave propagation constant, $j = (-1)^{1/2}$, x is the distance from the surface (depth), z is the direction of wave propagation and ω is angular frequency.

2.1.2. Boundary conditions

Mechanical field generated by Love waves propagating in an inhomogeneous elastic graded medium satisfies the following boundary conditions:

(a) on a free surface ($x = 0$), the transverse shear stress $\tau_{yx} = c_{44}(0) \cdot \frac{dV(0)}{dx} \cdot exp\{j(\beta z - \omega t)\}$ is equal to zero, hence $\frac{dV(0)}{dx} = 0$
(b) at each interface between two layers the condition of continuity of mechanical displacement v and transverse shear stress τ_{yx} is fulfilled
(c) at large distances ($x \to \infty$) from the surface ($x = 0$) the mechanical displacement of the Love wave should tend to zero, i.e., $V(\infty) = 0$.

2.1.3. Governing equations

The equation of motion (along with the appropriate boundary conditions) for Love waves propagating in an inhomogeneous elastic medium (isotropic and in some specified directions in media of regular and hexagonal symmetry) is represented by the following Differential Problem:

$$\frac{d}{dx}\left(c_{44}(x)\frac{dV(x)}{dx}\right) + \rho\omega^2 V(x) = c_{44}(x)\beta^2 V(x) \quad (1)$$

$$\frac{d}{dx}V(0) = 0, \quad V(\infty) = 0 \quad (2)$$

where $V(x)$ is the mechanical displacement distribution of the Love wave with the depth x. $c_{44}(x)$ is the elastic shear modulus that depends on the depth. β^2 is an eigenvalue determining the phase velocity of the Love wave. ρ is the density of the medium, and ω is the angular frequency.

The Sturm–Liouville Differential Problem (1–2) is a mathematical model describing the propagation of SH surface Love waves in nonhomogeneous elastic graded materials.

The solution of this Direct Sturm–Liouville Problem is a set of pairs $\{\beta_i^2, V_i(x)\}$; wherein β_i^2 is the i-th eigenvalue, $i = 0, 2, \ldots, n_1 - 1$, n_1 is the number of modes of Love waves propagating in considered waveguide and $V_i(x)$ is the eigenvector corresponding to this eigenvalue. Eigenvalue corresponds to the phase velocity of the SH surface wave, while the eigenvector describes the distribution of the mechanical displacement of the corresponding mode of the SH surface wave as a function of depth.

For a given frequency of the SH surface wave we obtain the corresponding eigenvalue i.e., Love wave phase velocity. The resulting set of the of phase velocities of the Love surface wave for various values of frequency determines the dispersion curve of the Love wave.

The concentration of the Love wave energy in the vicinity of the surface for the fundamental mode ($i = 0$) is large. The concentration of energy in the subsurface region for higher modes is much smaller (due to their large penetration depth). For this reason, sensors of physical quantities of liquids (e.g., viscosity and density) based on the use of the fundamental mode of Love waves have the highest sensitivity.

Higher modes of Love waves propagate at higher frequencies than the fundamental mode. In the actual media, at higher frequencies the losses are greater. Losses in the viscoelastic media grow with the square of the frequency. For this reason, higher modes of Love waves are subject to a stronger attenuation than the fundamental mode, as they travel.

All the above mentioned properties of the Love waves motivate the use only fundamental mode in the Love wave based chemo and bio-sensors.

Considering the above arguments, in the present paper, we restricted our analysis of the propagation of Love waves in graded materials to the fundamental mode ($i = 0$) of Love waves. The constant density of the considered graded materials $\rho = \rho_0 = const$ was assumed throughout the paper.

2.2. Group velocity

The group velocity of Love wave was calculated by means of a method employed in the theory of planar optical waveguides [29]. A similar relationship between the phase velocity and the group velocity can be developed by using formulas for the potential and kinetic energy of Love waves resulting from Analytical Mechanics [30]. These two methods are integral methods, in which information about the eigenvalue (phase velocity of the Love wave v_p) and the eigenvector (distribution of the mechanical displacement $V(x)$ of the Love wave with the depth) is used.

The Differential Problem Eqs. (1) and (2) can be formulated in integral (variational) form in terms of the Rayleigh quotient:

$$\beta^2 = \frac{\int_0^\infty \left[-c_{44}(x)\left(\frac{dV(x)}{dx}\right)^2 + \rho\omega^2 V^2(x)\right]dx}{\int_0^\infty c_{44}(x)V^2(x)dx} \quad (3)$$

By differentiating the Rayleigh quotient (Eq. (3)) with respect to the angular frequency ω [28], we arrive at the following formula:

$$\frac{v_p v_g}{v_0^2} = \frac{\int_0^\infty \frac{c_{44}(x)}{c_0} V^2(x)dx}{\int_0^\infty V^2(x)dx} \quad (4)$$

where c_0 is the shear elastic modulus in the substrate, and $v_0 = \sqrt{c_0/\rho_0}$ is the phase velocity of bulk SH waves in the substrate, ($x \to \infty$).

Eq. (4) links v_p and v_g for Love waves propagating in elastic graded materials. Knowing v_p (for given values of v_0 and $V(x)$), one can calculate the group velocity v_g and vice versa.

3. Various profiles in graded materials

Profiles of elastic properties of the surface layers in the graded materials are produced due to the use of various technological processes such as rolling, laser hardening (parabolic profile), shot peening, nitriding, carburizing (linear profile), boronizing. Moreover, the processes typical for the microelectronics and integrated optics, such as ion implantation and diffusion lead to exponential and Gaussian profiles.

In the present study the following profiles of elastic properties (shear modulus $c_{44}(x)$) in heterogeneous graded materials were examined, see Fig. 2a and b:

1. Profiles of the power-law type
   (a) square root type profile $n = 1/2$ (profile no. 1 in Fig. 2a)

$$c_{44}(x)/c_0 = 1 - (\Delta c/c_0)[1 - (x/D)^{1/2}][H(x - D) - H(x)] \quad (5a)$$

   (b) linear profile $n = 1$ (profile no. 2 in Fig. 2a)

$$c_{44}(x)/c_0 = 1 - (\Delta c/c_0)[1 - x/D][H(x - D) - H(x)] \quad (5b)$$

   (c) quadratic profile $n = 2$ (profile no. 3 in Fig. 2a)

$$c_{44}(x)/c_0 = 1 - (\Delta c/c_0)[1 - (x/D)^2][H(x - D) - H(x)] \quad (5c)$$

   (d) power type profile $n = 10$ (profile no. 4 in Fig. 2a)

$$c_{44}(x)/c_0 = 1 - (\Delta c/c_0)[1 - (x/D)^{10}][H(x - D) - H(x)] \quad (5d)$$

   (e) step profile $n = \infty$ (typical for classical Love wave, profile no. 5 in Fig. 2a)

$$c_{44}(x)/c_0 = 1 - (\Delta c/c_0)[H(x - D) - H(x)] \quad (5e)$$

where $H(x)$ is the Heaviside step function, D is the depth of an inhomogeneous elastic layer.

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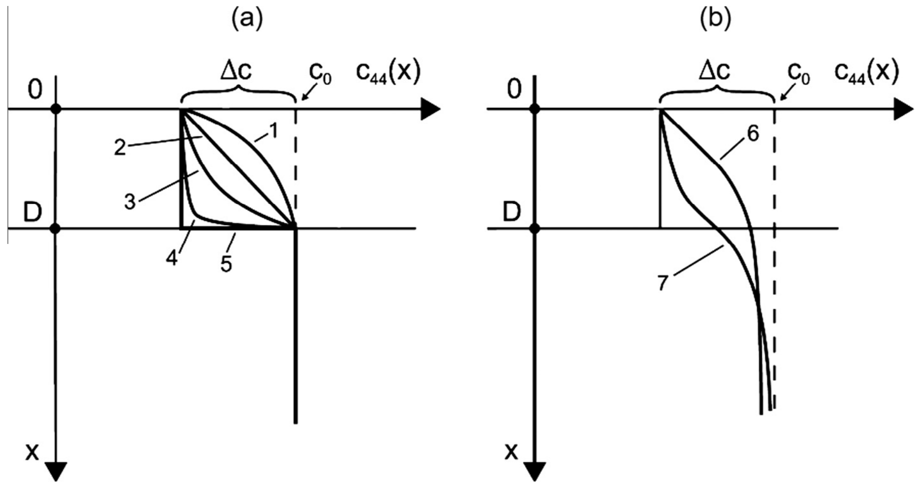

Fig. 2. (a, b) Considered profiles of elastic properties of graded materials, (a) profiles in a nonhomogeneous surface layer $x \in [0, D]$, (b) profiles in a nonhomogeneous half-space $x \in [0, \infty]$.

2. Exponential profile (profile no. 6 in Fig. 2b)

$$c_{44}(x)/c_0 = 1 - (\Delta c/c_0) \cdot \exp(-2x/D) \tag{5f}$$

3. Profile of the $1/\cosh^2(x)$ type, which is similar to the Gaussian profile (profile no. 7 in Fig. 2b)

$$c_{44}(x)/c_0 = 1 - (\Delta c/c_0) \cdot 1/\cosh^2(2x/D) \tag{5g}$$

The profiles of the elastic parameter $c_{44}(x)$ from Fig. 2a and b are realistic and physically realizable.

The surface SH Love wave propagates in a nonhomogeneous elastic graded material. The elastic properties of this medium $c_{44}(x)$ are only a function of the depth x. In the case of square root, linear, quadratic, power ($n = 10$), and step profiles the inhomogeneity in the elastic constant has a finite depth D. On the other hand, for exponential and $1/\cosh^2(2x/D)$ type profiles the disturbance of the elastic constant $c_{44}(x)$ extends to infinity ($x \to \infty$). The parameter D in formulas (5f) and (5g) specifies the depth of the disturbed surface layer.

4. Solution of the Direct Sturm–Liouville Problem

No analytical solutions for the Differential Problem (1–2) are available for general depth dependence of variable coefficient $c_{44}(x)$. Solution of the Sturm–Liouville Problem (1–2) for arbitrary function $c_{44}(x)$ is possible only numerically. A variety of numerical methods to solve this kind of differential problems exist, namely:

- Finite Difference Method (FDM).
- Variational Methods, e.g., the Ritz Method.
- Galerkin Method.
- Finite Element Method (FEM).
- Boundary Element Method (BEM).
- Green Function Method.
- Transfer Matrix Method.

The Differential Problem (1–2) is one-dimensional, therefore the above-mentioned numerical methods are essentially equivalent in application to solve this Differential Problem (1–2).

In the present work to solve the Differential Problem (1–2) two numerical methods, i.e., (1) Finite Difference Method (FDM), and (2) Transfer Matrix Method have been used. Solution of the Direct Sturm–Liouville Problem can constitute the basis for the solution of the Inverse Sturm–Liouville Problem.

4.1. Finite Difference Method (FDM)

Finite Difference Method is the simplest numerical method for solving differential boundary problems and for one-dimensional problems is equivalent to the Finite Element Method. The unknown function of mechanical displacement $V = [V_1, \ldots, V_m]^T$ is calculated only for a finite number m of discrete points in the region of interest in the x-axis, $[0, H]$. Region of interest $[0, H]$ was equal $10 \times D$ for both cases of discretization shown in Fig. 3a and b. In this work, the interval $[0, H]$ was discretized over a mesh of $m = 100$ equally distributed points.

Differential operator from Eq. (1) is approximated by a Difference Operator. In this manner, we obtain a Difference Problem, which approximates the initial Differential Problem (1–2):

$$[M][V] = \beta^2 [B][V] \tag{6}$$

where $V = [V_1, \ldots, V_m]^T$ is the eigenvector, β^2 is the eigenvalue. The matrix $[M]$ is a tridiagonal matrix of dimension ($m \times m$). The matrix $[B]$ is a diagonal matrix ($m \times m$).

By solving the matrix eigen-equation (6) we get a set of pairs $\{\beta_i^2, V_i\}$; where β_i^2 is the i-th eigenvalue, and $V_i = [V_1^i, \ldots, V_m^i]^T$ is the eigenvector corresponding to the i-th eigenvalue. The largest positive eigenvalue $\beta_i^2 = \beta_{max}^2$ corresponds to the fundamental mode of the Love wave propagating in a considered nonhomogeneous graded medium. In this manner, we obtain the phase velocity of the Love wave $v_p = \omega/\beta_{max}$. The corresponding eigenvector $V_{max} = [V_1^{max}, \ldots, V_m^{max}]^T$ determines the distribution of the mechanical displacement of the Love wave with depth.

4.2. Transfer Matrix Method

The equation of motion Eq. (1) is an ordinary differential equation of second order. By introducing a new variable $T = c_{44}(x)\frac{dV}{dx}$, this second order differential Eq. (1) can be represented as a system of two differential equations of the first order, namely:

$$\frac{d}{dx}\begin{bmatrix} V \\ T \end{bmatrix} = [A]\begin{bmatrix} V \\ T \end{bmatrix} = \begin{bmatrix} 0, & \frac{1}{c_{44}(x)} \\ \beta^2 c_{44}(x) - \omega^2 \rho(x), & 0 \end{bmatrix}\begin{bmatrix} V \\ T \end{bmatrix} \tag{7}$$

Transverse shear stress can be represented as: $\tau_{yx} = T(x) \cdot \exp[j(\beta z - \omega t)]$. The system of Eq. (7) will be solved by using the Transfer Matrix Method.

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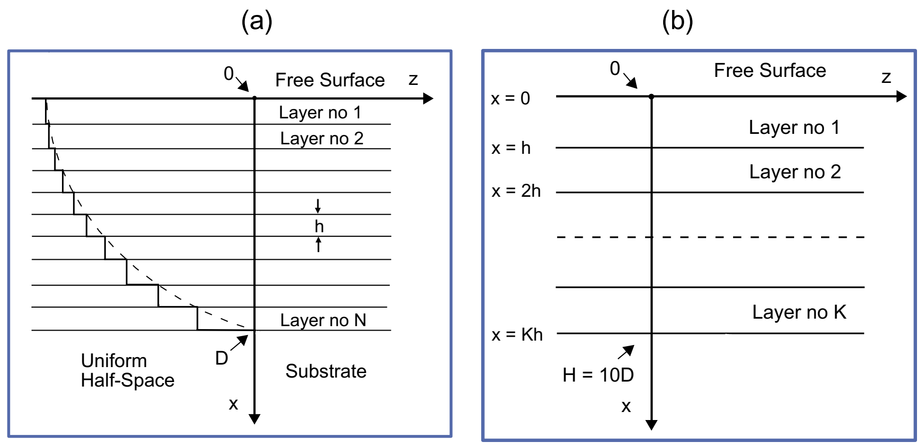

Fig. 3. (a, b) Partition of a nonhomogeneous Love wave waveguide into homogeneous elastic layers, (a) for power type profiles (curves nos. 1, 2, 3, 4, and 5 in Fig. 2a), $N = 10$, and (b) for the exponential profile and the profile of $1/\cosh^2$ type (curves nos. 6, and 7 in Fig. 2b), $K = 100$.

Non-homogeneous profiles of the modulus $c_{44}(x)$, that were considered in this work are shown in Fig. 2a and b. In the Transfer Matrix Method, non-homogeneous elastic graded medium, in which the Love wave propagates, is divided into a finite number of homogeneous elastic layers. The non-homogeneous elastic layer (rigidly attached to the homogeneous substrate) from Fig. 2a is divided into N parts, see Fig. 3a. On the other hand non-uniform elastic half-space from Fig. 2b is divided into K equal parts, see Fig. 3b.

At the interfaces between the subsequent elastic layers, continuity condition of the mechanical displacement V and transverse shear stress T is fulfilled. Eq. (7) is a matrix differential equation. The solution of this equation is as follows:

$$\begin{bmatrix} V(x) \\ T(x) \end{bmatrix} = [\exp(Ax)] \cdot \begin{bmatrix} V(0) \\ T(0) \end{bmatrix} \tag{8}$$

The matrix $[A]$ can be diagonalized i.e., can be represented as $[A] = [U][D][U^{-1}]$, where $[D]$ is a diagonal matrix, the diagonal of which contains eigenvalues (λ_1, λ_2) of matrix $[A]$ i.e., $\lambda_1 = \nu = \sqrt{\beta^2 - \omega^2\rho(x)/c_{44}(x)}$ and $\lambda_2 = -\nu = -\sqrt{\beta^2 - \omega^2\rho(x)/c_{44}(x)}$.

$[U]$ is a 2×2 matrix, the columns of which are eigenvectors of the matrix $[A]$, thus we can write:

$$[U] = \begin{bmatrix} 1, & 1 \\ \nu, & -\nu \end{bmatrix}; \quad [U^{-1}] = \frac{1}{2\nu}\begin{bmatrix} \nu, & 1 \\ \nu, & -1 \end{bmatrix}; \quad [D] = \begin{bmatrix} \nu, & 0 \\ 0, & -\nu \end{bmatrix};$$

Consequently, using the formula of linear algebra we get:

$$\begin{aligned} [\exp(Ax)] &= [U] \cdot [\exp(Dx)] \cdot [U^{-1}] \\ &= \begin{bmatrix} 1, & 1 \\ \nu, & -\nu \end{bmatrix} \begin{bmatrix} \exp(\nu x), & 0 \\ 0, & \exp(-\nu x) \end{bmatrix} \frac{1}{2\nu} \begin{bmatrix} \nu, & 1 \\ \nu, & -1 \end{bmatrix} \end{aligned} \tag{9}$$

In general, the eigenvalues $(\nu, -\nu)$ are imaginary quantities, therefore we can write $\nu = jq$.

By performing matrix multiplication in formula (9) we arrive at the following form of matrix $[\exp(Ax)]$:

$$[\exp(Ax)] = \cos(qx) \begin{bmatrix} 1, & \frac{1}{c_{44}(x)q}\tan(qx) \\ -c_{44}(x)q\tan(qx), & 1 \end{bmatrix} \tag{10}$$

The matrix $[\exp(Ax)]$ is responsible for the transformation of the vector $[V, T]^T$ between the upper and the lower surface of a homogeneous elastic layer of thickness h, see Fig. 3a and b. Knowing the value of the displacement and stress $[V, T]^T$ in the plane $x = 0$, and using formulas (8) and (10), one can specify the value of the displacement and stress in the plane $x = h$, see Fig. 3a and b:

$$\begin{bmatrix} V \\ T \end{bmatrix}\Bigg|_{x=h} = \cos(qh) \cdot \begin{bmatrix} 1, & \frac{1}{c_{44}(h)q}\tan(qh) \\ -c_{44}(h)q\tan(qh), & 1 \end{bmatrix} \cdot \begin{bmatrix} V \\ T \end{bmatrix}\Bigg|_{x=0} \tag{11}$$

Applying this procedure successively from the last layer with the number K and/or N to the first layer with the number 1, see Fig. 3a and b, the mechanical displacement V and stress T in the plane $x = 0$ (upper limit of the region of interest) become dependent on the displacement and stress in the plane $x = H$ (lower limit of the region of interest), namely:

$$\begin{bmatrix} V \\ T \end{bmatrix}\Bigg|_{x=0} = \begin{bmatrix} P_{11}, & P_{12} \\ P_{21}, & P_{22} \end{bmatrix} \cdot \begin{bmatrix} V \\ T \end{bmatrix}\Bigg|_{x=H} \tag{12}$$

Matrix $[P]$ is the propagator matrix. Matrix $[P]$ is a product of (K and/or N) subsequent matrices (see Eq. (11)) contributed by each layer.

Insertion of the boundary conditions in the plane $x = 0$ and $x = H$ to Eq. (12) leads to the following nonlinear algebraic equation for β^2 as the unknown:

$$P_{22}(\beta^2, \omega) = 0 \tag{13}$$

Eq. (13) is the dispersion equation for the Love wave propagating in the considered heterogeneous graded media.

Solving this equation (for a given value of ω) we obtain the value of the wave number β. From the knowledge of the wave number β one can then calculate the phase velocity of the Love wave $v_p = \omega/\beta$.

Presented above procedure constitute the basis of the Transfer Matrix Method [31].

The Transfer Matrix Method will be used in the numerical calculations as one of methods for solving the Direct Sturm–Liouville Problem.

5. Results of numerical calculations and discussion

To study the propagation behavior of ultrasonic Love waves in non-homogeneous graded materials from Fig. 2a and b, the following material parameters are assumed in the numerical calculations:

$$c_0 = 2.564 \times 10^{10}\,\frac{N}{m^2}; \quad v_0 = 1849\,\frac{m}{s};$$
$$\rho_0 = 7.5 \times 10^3\ \text{kg/m}^3; \quad \Delta c/c_0 = 0.088; \quad D = 0.4\ \text{mm}.$$

These parameters are typical for PZT-4 ceramics [32] with elastic properties perturbed in the vicinity of the surface.

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Applying the Transfer Matrix Method and the Finite Difference Method the dispersion curves of Love waves propagating in the considered graded materials presented in Fig. 2a and b were evaluated.

The variation of the phase velocity of the Love wave that propagates in graded materials from Fig. 2a and b, versus normalized depth D/L_0 (normalized frequency) is plotted in Fig. 4. Numbers of the dispersion curves in Fig. 4 correspond to the numbers of the elastic profiles marked in Fig. 2.

Fig. 5 displays the dispersion curves of phase velocity v_p and group velocity v_g of the Love wave propagating in the graded materials of the exponential profile (Eq. (5f)) given in Fig. 2b (plot no. 6).

Fig. 6 exhibits a graph of phase velocity v_p and group velocity v_g of Love waves propagating in a graded medium with the profile of the $1/\cosh^2(2x/D)$ type, see Eq. (5g).

5.1. Influence of the inhomogeneity coefficient on the dispersion curves

The analysis of the effect of the inhomogeneity coefficient α on the Love wave dispersion curves was performed on the example of the $1/\cosh^2(\alpha x)$ type profile. Profiles of changes in the elastic coefficient $c_{44}(x) = c_0 - \Delta c/\cosh^2(\alpha x)$, for the three values of the inhomogeneity coefficient: $\alpha = 1/D, 2/D, 4/D$, have been considered. Phase and group velocity dispersion curves of the Love wave propagating in the considered above waveguide structures have been calculated, see Figs. 7 and 8.

As can be seen from Figs. 7 and 8, a change in the inhomogeneity coefficient α has a remarkable effect on the phase and group velocity dispersion curves of Love waves. Love waves in the high-frequency and low frequency range (for higher and lower values of normalized frequency D/L_0) are less sensitive to the changes of the inhomogeneity coefficient than Love waves in the intermediate-frequency range (for moderate values of D/L_0).

5.2. Comparison of the results obtained from FDM and TMM

The calculation of the phase velocity dispersion curves were carried out by using two numerical methods, i.e., Finite Difference

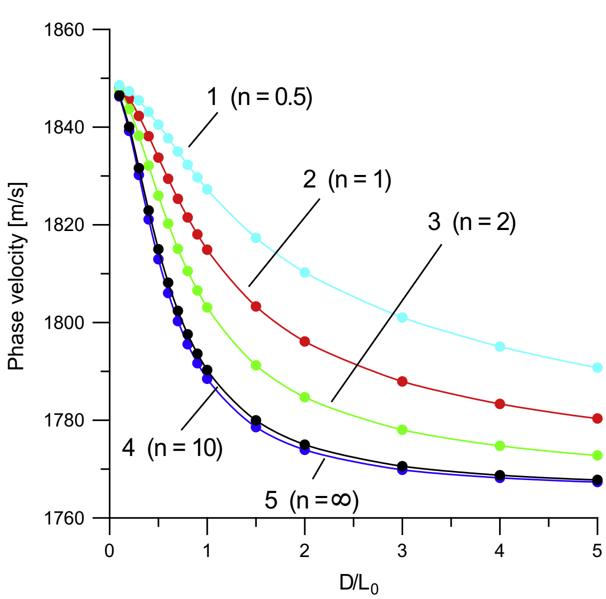

Fig. 4. Phase velocity v_p of Love waves propagating in heterogeneous graded materials presented in Fig. 2a. D is the thickness of the nonhomogeneous graded elastic layer, L_0 is the wavelength of the shear bulk wave in the substrate ($x \to \infty$).

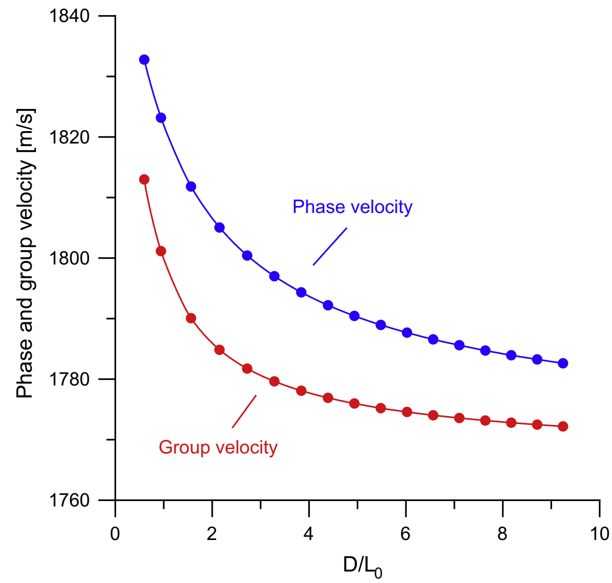

Fig. 5. Love wave dispersion curves of phase v_p and group v_g velocity for the shear modulus profile that follows the exponential profile (plot no. 6 in Fig. 2b).

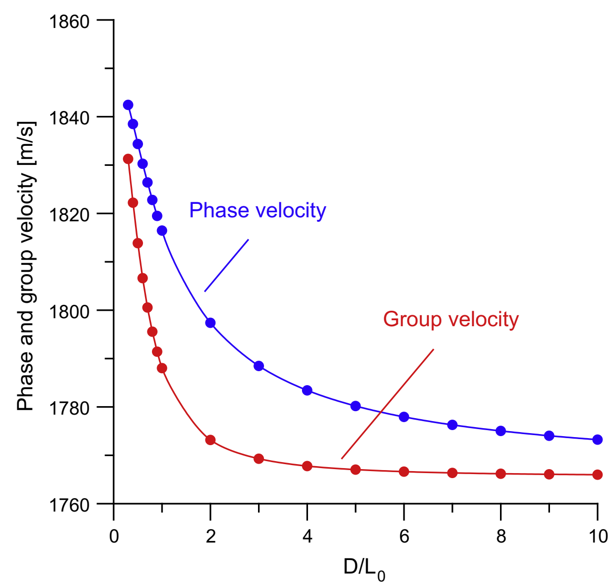

Fig. 6. Love wave dispersion curves of phase velocity v_p and group velocity v_g in the case of the profile of the $1/\cosh^2(2x/D)$ type (plot no. 7 in Fig. 2b).

Method and Transfer Matrix Method. Resulting from the FDM matrix Eq. (6) on the eigenvalues and eigenvectors, as well as the nonlinear algebraic Eq. (13) resulting from the Transfer Matrix Method for the eigenvalues β^2 have been solved by using numerical procedures of the software package Scilab.

Numerical calculations were performed for $h/D = 0.1$, where h is the thickness of a single layer obtained as a result of discretization of the region of interest on a finite number of homogeneous layers, see Fig. 3a and b. Only in the case of the exponential profile shown in Fig. 5, numerical calculations (for $h/D = 0.1$) were unstable. For this reason to eliminate the instability for exponential profile, the use of a smaller value of the relative layer thickness $h/D = 0.05$ was required.

From the numerical calculations of authors follows that the results obtained using these two methods are identical (with

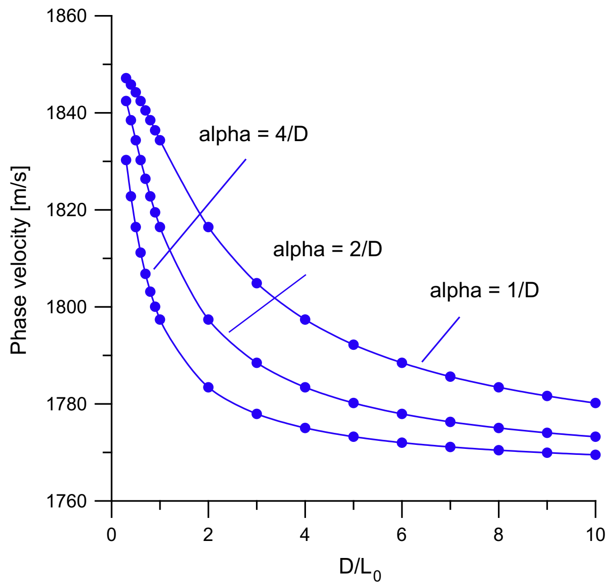

Fig. 7. Phase velocity dispersion curves of Love waves for different values of the inhomogeneity coefficient α for a profile of the $1/\cosh^2(\alpha x)$ type.

accuracy of 5 decimal places). This indicates that these two numerical methods are essentially equivalent in applications to describe the propagation of Love waves in considered elastic graded materials.

5.3. Discussion

As shown in Fig. 4 with the increase in the exponent n the phase velocity dispersion curves for power type profiles approach the classical Love wave dispersion curve for step profile ($n = \infty$). The penetration depth of the mechanical displacement of the Love wave (eigenvector) for step profile ($n = \infty$) is the lowest, and augments for the case of other power type profiles with decreasing values of the exponent n.

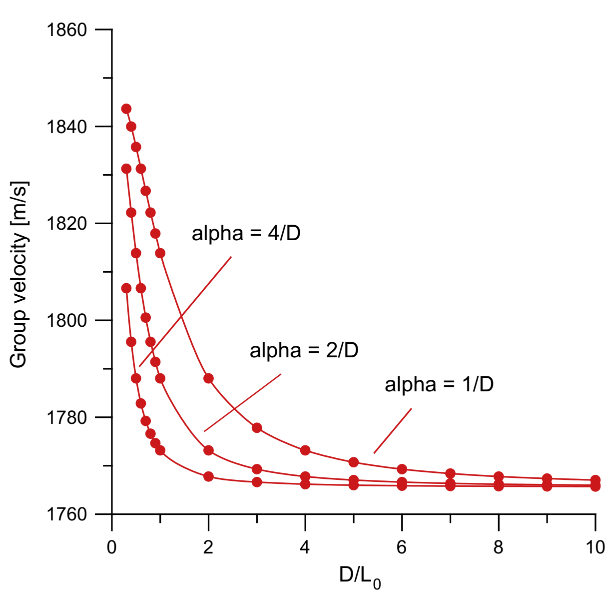

Fig. 8. Group velocity dispersion curves of Love waves for different values of the inhomogeneity coefficient α for a profile of the $1/\cosh^2(\alpha x)$ type.

As follows from Figs. 4–6, Love wave phase velocity tends to a value of the phase velocity of the bulk shear wave at ($x = 0$) with increasing D/L_0 (i.e., with the frequency increase). As can be seen from Figs. 5 and 6, the Love wave waveguide with the profile of the $1/\cosh^2$ type has better waveguide properties (higher slope of the dispersion curve in the region of the steepest descent) than the waveguide with the exponential profile. This indicates higher sensitivity of phase velocity to changes in frequency. This is of great importance in the design and construction of sensors of the physical parameters that use the Love wave.

Group velocity of the Love wave propagating in the graded materials with the exponential profile and the profile of the type $1/\cosh^2$ was determined using the integral formula (4).

As can be seen in Figs. 5–8 the group velocity dispersion curves of Love waves are always located below the corresponding phase velocity dispersion curves.

It is a characteristic property of media exhibiting the so-called normal dispersion.

In the numerical calculations, we assumed the value of thickness $D = 0.4$ mm. In this case, the value of $D/L_0 = 1$ corresponds to frequency 4.6225 MHz, and $D/L_0 = 10$ corresponds to frequency 46.225 MHz. In the frequency range considered by the authors ($4.6225 - 46.225$ MHz), higher modes of Love waves may also propagate. In this case, the largest positive eigenvalue (β^2) corresponds to the fundamental mode. Other possible positive eigenvalues that correspond to the subsequent higher modes are smaller. For example, for step profile (Eq. (5e)) and $f = 18.49$ MHz, i.e., $(D/L_0) = 4$, the fundamental mode ($\beta^2 = 4.32 \times 10^9$), the first overtone ($\beta^2 = 4.22 \times 10^9$), and second overtone ($\beta^2 = 4.04 \times 10^9$) occur. Based on this criterion, the fundamental mode was easily separated and discriminated from the possible higher modes.

Measurement of surface waves velocity in the MHz frequency range are usually performed in quantitative nondestructive evaluation experiments (QNDE). Hence, the results obtained in this paper can be important to the interpretation of experimental dispersion curves of surface Love waves propagating in elastic graded materials.

6. Conclusions

In this paper, a theoretical analysis of the propagation behavior of ultrasonic Love waves in functionally graded materials with the monotonic change in the elastic properties as a function of distance from the (treated) surface of the material is presented.

The problem of Love wave propagation in graded materials are formulated as a Direct Sturm–Liouville Problem. Within this Sturm–Liouville Problem, equations of motion of nonhomogeneous elastic medium with the appropriate boundary conditions were formulated. Direct Sturm–Liouville Problem was solved using two numerical methods, i.e., (1) Finite Difference Method, and (2) Transfer Matrix Method (Haskell-Thompson). Dispersion curves (velocity versus frequency) of the phase and group velocity of the Love wave propagating in the above mentioned graded materials were evaluated, what is a novelty. Very good agreement of dispersion curves obtained by these two numerical methods was stated. The integral formula for the group velocity of Love waves in nonhomogeneous elastic graded materials has been established, which is a novelty.

These two numerical methods are equivalent if the number of layers is relatively small (less than a few hundred). In contrast, for a large number of layers (over one thousand) Transfer Matrix Method is more efficient than Finite Difference Method. The effect of elastic properties inhomogeneity on the dispersion curves of Love waves was examined. It has been found that the profiles of the elastic properties similar to the step profile (the typical profile

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for classical Love wave propagating in a homogeneous layered media) exhibit the best waveguide properties (i.e., the slope of the dispersion curve is the highest). This is very important for applications of Love waves in sensors (e.g., in the bio and chemosensors and in sensors of physical properties of materials).

The results of this work may also find application in geophysics, seismology and underground acoustics to investigate the internal structure of Earth (crustal and subcrustal region near the Earth surface). These results can be also applied in exploration of natural resources (e.g., mineral oils, gases and minerals) [27]. Moreover, Love waves may also be used to investigate planar optical waveguides [28], and layered sensors and resonators of the MEMS (Micro Electro Mechanical Systems) type [3,25].

The results obtained in this paper are fundamental and can promote a better understanding of the behavior of Love wave propagating in elastic nonhomogeneous media (e.g., in functionally graded materials and composites).

According to the authors' knowledge the results obtained in this work are original and have not been reported in the scientific literature.

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