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Inverse method for evaluation of elastic parameters in functionally graded materials using ultrasonic Love wave

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Abstract— **The aim of this study was to evaluate the inverse procedure to determine profiles (as a function of depth) of the mechanical properties of inhomogeneous FGM resulting from the application of various technological processes of surface treatment. First, the Direct Sturm-Liouville Problem for Love waves propagating in elastic graded materials with various profiles of the shear stiffness as a function of the distance from the surface, has been solved using the Finite Difference Method and Transfer Matrix Method (Haskell-Thompson method). Love wave dispersion curves were evaluated in the frequency range from 4 to 23 MHz. The Inverse Problem was formulated as an Optimization Problem with appropriately constructed objective function that depended on the material properties of an elastic waveguide of the Love wave and the experimental data. To minimize the considered objective function, optimization procedures of the Nelder-Mead type from Scilab software package were employed.**

Keywords—Love waves; inverse problem; Sturm-Liouville problem; dispersion curves

I. INTRODUCTION

This paper presents the use of SH (Shear Horizontal) surface Love waves to determine the distributions of elastic parameters in inhomogeneous Functionally Graded Materials (FGM). The advantage of Love waves (applied to investigate the elastic properties of materials) in relation to the surface Rayleigh waves is that they have only one component of the mechanical displacement, in contrast to Rayleigh waves, which have two components [1]. Recently, ultrasonic method have been introduced to measure the physical properties of materials [2-7]. Love wave energy (in contrast to the other types of waves, e.g., plate Lamb waves) is concentrated in the vicinity of the surface layer. The penetration depth of the SH surface Love waves depends on the frequency. Therefore, Love waves are particularly suitable for investigating the profiles of the mechanical properties in inhomogeneous Graded Materials.

Direct Sturm-Liouville Problem that describes the propagation of Love waves in inhomogeneous graded materials has been formulated and solved numerically. The Inverse Procedure (Inverse Sturm-Liouville Problem) for determining the distribution of elastic properties versus depth in the inhomogeneous materials has been developed. Love wave dispersion curves in inhomogeneous graded materials were evaluated numerically (synthetic data). Using the evaluated dispersion curves of Love waves and a developed Inverse Procedure the distributions of elastic shear coefficient as a function of depth (distance from the surface of the material into the bulk) in a heterogeneous surface layer deposited on a homogeneous substrate have been evaluated. Formulation and solution of the Direct Problem and Inverse Problem for the Love wave propagating in the considered waveguide structures (see Fig. 1), in which the elastic properties vary continuously with depth, is a novelty. The results of this work can be applied in the investigation of Graded Materials applied in the aviation, aerospace and electronic industry, in fine mechanics, as well as in geophysics and seismology [8].

II. DIRECT STURM-LIOUVILLE PROBLEM

Consider the propagation of Love waves in inhomogeneous elastic half-space in which the shear elastic coefficient is a continuous function of depth, see Fig.1.

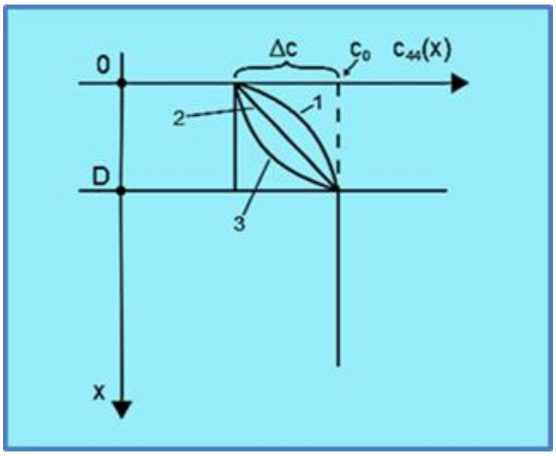

Fig.1. Variation of the elastic coefficient $c_{44}(x)$, as a function of depth, in a nonhomogeneous elastic graded layer deposited on a homogeneous elastic substrate.

Such structures may represent elastic media occurring in the structures used, among others, in the electronics, aerospace and astronautic industry.

Profiles of changes in the elastic coefficient $c_{44}(x)$ in a nonhomogeneous elastic layer deposited on a homogeneous surface (see Fig.1) are represented by the following formulas:

a) square root type profile $n = 1/2$ (profile no 1 in Fig.1)

$$c_{44}(x)/c_0 = 1 - (\Delta c/c_0)\left[1 - (x/D)^{1/2}\right][H(x - D) - H(x)] \quad (1a)$$

b) linear profile $n = 1$ (profile no 2 in Fig.1)

$$c_{44}(x)/c_0 = 1 - (\Delta c/c_0)\,[1 - x/D][H(x - D) - H(x)] \quad (1b)$$

c) quadratic profile $n = 2$ (profile no 3 in Fig.1)

$$c_{44}(x)/c_0 = 1 - (\Delta c/c_0)\,[1 - (x/D)^2][H(x - D) - H(x)] \quad (1c)$$

where: $H(x)$ is the Heaviside step function, D is the depth of an inhomogeneous elastic layer.

Shear horizontal surface Love waves propagating in the considered heterogeneous elastic waveguide can be represented in the following form [9]:

$u = f(x)\exp j(\beta z - \omega t)$, where: $f(x)$ is the amplitude of the Love wave, β is a propagation constant of the wave, $j = (-1)^{1/2}$, x is the distance from the surface (depth), z is the direction of wave propagation and ω is an angular frequency. Love wave mechanical movement is performed along the y axis.

Mechanical field generated by Love waves propagating in an inhomogeneous elastic graded medium satisfies the following boundary conditions:

a) on a free surface ($x = 0$), the transverse shear stress

is equal to zero, hence $\frac{df(0)}{dx} = 0$

b) at large distances ($x \to \infty$) from the surface ($x = 0$) the mechanical displacement of the Love wave should tend to zero, i.e., $f(\infty) = 0$.

The equation of motion for Love waves propagating in an inhomogeneous elastic medium (isotropic and in certain specified directions in media with regular and hexagonal symmetry) is represented by the following Differential Problem:

$$\frac{d}{dx}\left(c_{44}(x)\frac{df}{dx}\right) + \rho\omega^2 f = c_{44}(x)\beta^2 f \quad (2)$$

$$\frac{df(0)}{dx} = 0 \quad ; \quad f(\infty) = 0 \quad (3)$$

The Differential Problem (2 and 3) is named the Direct Sturm-Liouville Problem. The solution of the Direct Sturm-Liouville Problem is a set of pairs $\left(\beta_j^2,\ f_j(x)\right)$; where: β_j^2 is the $j - th$ eigenvalue, $j = 1,2,\ldots,n$; n is the number of modes of Love waves propagating in considered waveguide and $f_j(x)$ is the eigenvector corresponding to this eigenvalue. Eigenvalue corresponds to the phase velocity of the propagating surface Love wave, while the eigenvector describes the distribution of the mechanical displacement of an appropriate mode of the surface wave as a function of depth.

The constant density of the considered graded materials $\rho = \rho_0 = const$ was assumed throughout the paper.

III. Inverse Sturm-Liouville Problem

Inverse Problem considered in this work relies on the determination of unknown elastic parameters from the measured dispersion curves (phase velocity as a function of frequency) for Love waves that propagate in an inhomogeneous elastic waveguide. To solve the Inverse Problem one has to perform the following steps:

- solve Direct Problem
- determine experimentally dispersion curves
- solve Inverse Problem.

In this paper, the Inverse Problem was formulated and solved as an optimization problem (Liu and Han, 2003) with properly defined objective function.

A. Objective function

The objective function Π depends on the distribution of the elastic coefficient $c_{44}(x)$ in nonhomogeneous investigated elastic structure, frequency, and experimental data (phase velocity of the surface Love wave). The nonhomogeneous elastic layer from Fig.1 was divided into 10 homogeneous layers. Values of the $c_{44}(x)$ coefficient at 9 evenly spaced discrete points of the surface layer, i.e., $c_{44}(x_j)$, $j = 1, 2, \ldots, 9$ are treated as the unknowns to be determined from the Inverse Procedure. The objective function was introduced and defined as:

$$\Pi(t_1, t_2, \ldots, t_9) = \sum_{j=1}^{N_e}\left\{\left(\frac{v_j^{exp} - v_j^{cal}(t_1, t_2, \ldots, t_9)}{v_j^{exp}}\right)^2\right\} \quad (4)$$

where: N_e - is the number of experimental frequencies, v_j^{exp} - is the measured phase velocity, v_j^{cal} - is the calculated phase velocity, $t_1 = c_{44}(x_1)$, $t_2 = c_{44}(x_2)$, … ,$t_9 = c_{44}(x_9)$ - represent operational variables, that are determined from the solution of the Inverse Sturm-Liouville Problem.

Making use of the optimization methods a minimum of the objective function was determined. This enabled the determination of the optimum values for the unknown distribution of the elastic coefficient $c_{44}(x)$ in the nonhomogeneous graded layer. To minimize the considered objective function Π the appropriate optimization procedures (of the Nelder-Mead type) from the Scilab software package were employed.

B. Numerical experiment (synthetic data)

In this study, the following values of material parameters of the considered nonhomogeneous elastic half-space from Fig.1 were used:

$c_0 = 2.564 \cdot 10^{10}\ N/m^2$; $v_0 = 1849\ m/s$;

$\rho_0 = 7.5 \cdot 10^3\ kg/m^3$; $\Delta c/c_0 = 0.088$.

These parameters are typical for PZT-4 ceramics with elastic properties perturbed in the vicinity of the surface.

In our numerical experiments we determined the dispersion curves numerically. In the first step, by solving the Direct Problem (Eqs.2 and 3), we calculated the phase velocity curves for the exact values of the elastic coefficient $c_{44}(x)$ from Fig.1. Exemplary dispersion curve represented in Fig.2 by solid line (not corrupted by noise) is regarded as an exact dispersion curve.

Subsequently, we added random errors (e.g., 1%, 5%, 10%) to these values of the phase velocity. Obtained in this way dispersion curves are treated in the Inverse Procedure as simulated experimental curves. Points marked by ♦ indicate in Fig.2 exemplary synthetic (experimental) dispersion curves obtained by corrupting the exact dispersion curves by random noise on the level of ±1%.

The upper and lower dashed lines delimit the synthetic dispersion curves of phase velocity which results from exact dispersion curves that are subject to corruption by ±1% maximum random noise. Dispersion curves were evaluated and disturbed by random noise for 6 various values of normalized frequency D/L: $D/L = 0.5, 1.0, 2.0, 3.0, 4.0$ and 5.0.

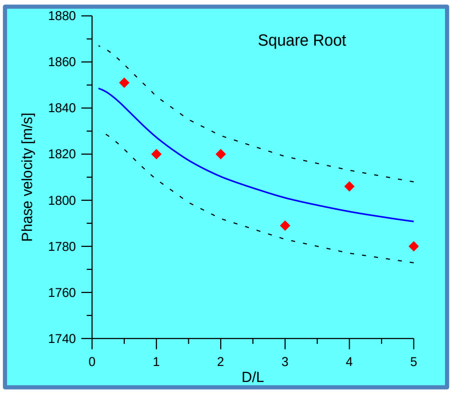

Fig.2. Phase velocity dispersion curves of Love wave propagating in a nonhomogeneous Graded elastic surface layer deposited on a homogeneous substrate. Elastic coefficient $c_{44}(x)$ in the surface layer varies according to the square root function of the depth.

IV. RESULTS OF NUMERICAL CALCULATIONS AND DISCUSSION

The Direct Problem that describes the propagation of Love waves in nonhomogeneous elastic Graded Materials was formulated and solved numerically by employing the Transfer Matrix Method [10]. Theoretical (exact) dispersion curves for the Love surface waves, propagating in the selected nonhomogeneous structures, were solutions of the Direct Problem.

The nonhomogeneous surface layer from Fig.1 $x \in [0, D]$ was divided into 10 homogeneous elastic layers. Unknown values of the elastic coefficient $c_{44}(x)$ are determined in 9 evenly spaced points $[x_1, x_2, \dots, x_9]$ at the layers' boundaries. Thus, an unknown vector of the elastic coefficient is sought in the form of: $c_{44}^{eval} = [c_{44}(x_1),\ c_{44}(x_2),\ \dots\ , c_{44}(x_9)]^T$.

Figure 3 illustrates an exemplary distribution of elastic coefficient c_{44}^{eval} in the surface layer as a function of depth for the square root type profile obtained using the Inverse Method. Numerical experiment has been conducted for random noise level of 1%.

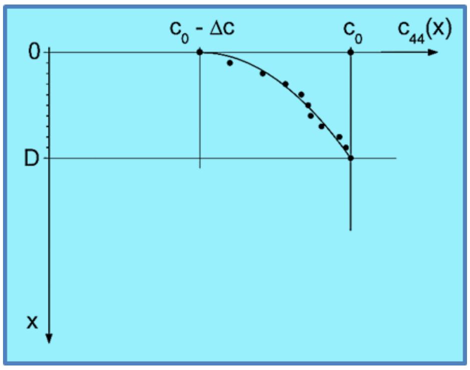

Fig.3. Elastic coefficient c_{44}^{eval} evaluated from the Inverse Problem (dotted line). Solid line represents an exact distribution of the shear elastic coefficient $c_{44}(x)$ for square root type profile (given by Eq.1a).

Exemplary distribution of changes in the elastic coefficient c_{44}^{eval} in the surface layer, resulting from the application of the Inverse Method, is presented in Fig.4. Numerical experiment has been performed for random noise level of 1% (linear profile).

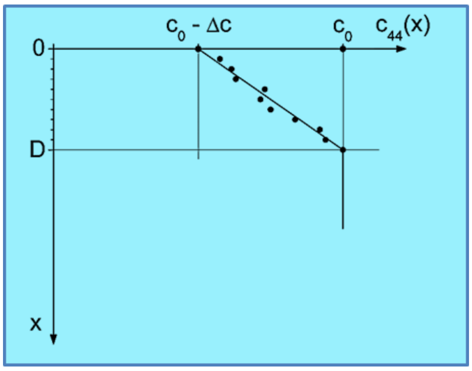

Fig.4. Elastic coefficient c_{44}^{eval} evaluated from the Inverse Problem (dotted line). Solid line represents an exact distribution of the shear elastic coefficient $c_{44}(x)$ for the linear type profile (given by Eq.1b).

Figure 5 shows (obtained from the Inverse Method) an exemplary distribution of the elastic coefficient c_{44}^{eval} in the surface layer as a function of depth for a quadratic type profile. Numerical experiment has been performed for random noise level of 1%.

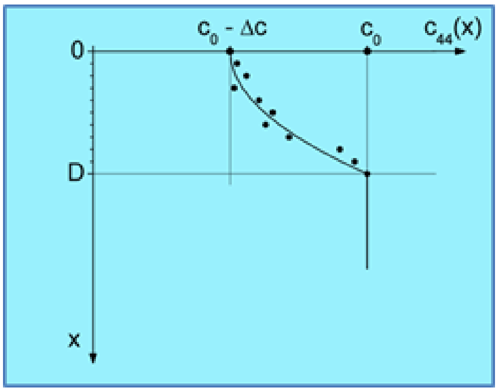

Fig.5. Elastic coefficient c_{44}^{eval} evaluated from the Inverse Problem (dotted line). Solid line represents an exact distribution of the shear elastic coefficient $c_{44}(x)$ for quadratic type profile (given by Eq.1c).

A. Relative error

Using the concept of norm $\|\cdot\|$ (introduced by the Polish mathematician Stefan Banach), relative error of a single measurement of the elastic coefficient $c_{44}(x)$ can be defined as follows: Relative Error = $\|c_{44}^{eval} - c_{44}^{exact}\|/\|c_{44}^{exact}\|$.

In this work, as the norm of the numerical sequence, the l_1 norm was chosen. This norm is the sum of modulus of subsequent sequence elements. In this way, the relative error ($R\ err$) of a single measurement (evaluation) of the elastic coefficient $c_{44}(x)$ amounts to:

$$(R\ err)_{N=1} = \frac{\|c_{44}^{eval} - c_{44}^{exact}\|_{l_1}}{\|c_{44}^{exact}\|_{l_1}} = \frac{|c_{44}^{eval}(x_1) - c_{44}^{exact}(x_1)| + |c_{44}^{eval}(x_2) - c_{44}^{exact}(x_2)| + \cdots + |c_{44}^{eval}(x_9) - c_{44}^{exact}(x_9)|}{|c_{44}^{exact}(x_1)| + |c_{44}^{exact}(x_2)| + \ldots + |c_{44}^{exact}(x_9)|} \quad (5)$$

Similarly, the relative error for a series of N evaluations (from the solution of the Inverse Problem) of the distribution of the elastic coefficient $c_{44}(x)$, is defined as follows:

$$(R\ err)_N = \left\{ \frac{\|(c_{44}^{eval})_1 - c_{44}^{exact}\|_{l_1}}{\|c_{44}^{exact}\|_{l_1}} + \frac{\|(c_{44}^{eval})_2 - c_{44}^{exact}\|_{l_1}}{\|c_{44}^{exact}\|_{l_1}} + \cdots + \frac{\|(c_{44}^{eval})_N - c_{44}^{exact}\|_{l_1}}{\|c_{44}^{exact}\|_{l_1}} \right\} / N \quad (6)$$

For subsequent profiles of the elastic modulus from Fig.1, a series of $N = 10$ numerical measurements of Love wave dispersion curves was conducted. To this end, using a random number generator, for each profile 10 different dispersion curves of Love waves were evaluated corrupting the exact dispersion curve by a the random error of a specific level. Each of these dispersion curves (synthetic data), was used in the calculations of the Inverse Method. Using, obtained in such a manner, elastic coefficient profiles (c_{44}^{eval}), the relative error of determining the distribution of the elastic coefficient $c_{44}(x)$, treated as a function of depth, has been determined, see Table I.

TABLE I.

Relative error of the determination of the elastic coefficient $c_{44}(x)$ evaluated from the Inverse Method, for the maximum random errors equal to 0.1%, 1%, 5%, and 10%. Each evaluation of the elastic coefficient (c_{44}^{eval}) results from the minimization of the objective function Π (Eq.4), for subsequent simulated dispersion curves of phase velocity.

Random error	0.1 %	1 %	5 %	10 %
(Relative error $)_{N=10}$ square root profile [%]	3.59	9.93	13.31	16.21
(Relative error $)_{N=10}$ linear profile [%]	4.58	9.39	13.52	15.42
(Relative error $)_{N=10}$ quadratic profile [%]	2.68	6.31	9.98	14.67

As can be seen from Table I and figures 3, 4 and 5, the proposed Inverse Method can be effectively used to identify the modulus of elasticity $c_{44}(x)$ profile changes in Graded Materials. The accuracy of the obtained (from Inverse Method) modulus of elasticity $c_{44}(x)$ profile changes is good.

V. CONCLUSIONS

An Inverse Method that uses Love surface waves for determining the distribution of the shear elastic coefficient $c_{44}(x)$ in elastic Functionally Graded Materials, from evaluated dispersion curves, has been developed.

In the paper, the Sturm-Liouville Direct Problem for the Love wave propagating in a nonhomogeneous elastic layer deposited on the homogeneous substrate was formulated and solved using the Transfer Matrix Method. Subsequently, the Inverse Problem for the ultrasonic Love wave propagating in the considered inhomogeneous waveguide structure was also formulated and solved. The Inverse Problem was formulated and solved as an optimization problem.

Formulation and solution of the Direct Problem and Inverse Problem for the Love wave propagating in the considered elastic graded structures is a novelty.

The results obtained in this study can be helpful in determining profiles of elastic coefficients changes in various Graded Materials. Materials of this type are produced during technological processes used in many industries such as: electronic, aviation, aerospace, automotive as well as in medicine and biomechanics. Moreover, the results of this work can also be employed in geophysics and seismology.

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