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Direct Sturm–Liouville problem for surface Love waves propagating in layered viscoelastic waveguides

P. Kiełczyński

Institute of Fundamental Technological Research, Polish Academy of Sciences, ul. Pawińskiego 5B, 02-106 Warsaw, Poland

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ABSTRACT

This paper presents theoretical model for shear-horizontal (SH) surface acoustic waves of the Love type propagating in lossy waveguides consisting of a lossy viscoelastic layer deposited on a lossless elastic half-space. To this end, a direct Sturm–Liouville problem that describes Love waves propagation in the considered viscoelastic waveguides was formulated and solved, what constitutes a novel approach to the state-of-the-art. To facilitate the solution of the complex dispersion equation, the Author employed an original approach that relies on the separation of its real and imaginary part. By separating the real and imaginary parts of the resulting complex dispersion equation for a complex wave vector $k = k_0 + j\alpha$ of the Love wave, a system of two real nonlinear transcendental algebraic equations for k_0 and α has been derived. The resulting set of two algebraic transcendental equations was then solved numerically. Phase velocity v_p and coefficient of attenuation α were calculated as a function of the wave frequency f, thickness of the surface layer h and its viscosity η_{44}. Dispersion curves for Love waves propagating in lossy waveguides, with a lossy surface layer deposited on a lossless substrate, were compared to those corresponding to Love surface waves propagating in lossless waveguides, i.e., with a lossless surface layer deposited on a lossless substrate. The results obtained in this paper are original and to some extent unexpected. Namely, it was found that: 1) the phase velocity v_p of Love surface waves increases as a function of viscosity η_{44} of the lossy surface layer, and 2) the coefficient of attenuation α has a maximum as a function of thickness h of the lossy surface layer. The results obtained in this paper are novel and can be applied in geophysics, seismology and in the optimal design and development of viscosity sensors, bio and chemosensors.

1. Introduction

By contrast to bulk acoustic waves, propagating in volume of solids, surface acoustic waves are guided by a free surface of the solid and their amplitude decays exponentially as a function of distance from the guiding surface. Existence of surface waves in solids was predicted theoretically in 1885 by Lord Rayleigh, who showed mathematically that such waves are solutions of a boundary value problem in an elastic half space [1]. Rayleigh surface waves in isotropic solids have two coupled components of particle vibrations, i.e., a longitudinal component L, which is parallel to the direction of wave propagation, and a transverse vertical (SV) component, which is perpendicular to the direction of wave propagation but is normal to the guiding surface.

E-mail addresses: pkielczy@ippt.gov.pl, pkielczy@ippt.pan.pl

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It is easy to show that an elastic surface wave with uniquely one SH component of particle vibrations (which is perpendicular to the direction of wave propagation and parallel to the guiding surface) cannot exist in a solid half space, since the corresponding boundary conditions on a free surface of the elastic half space cannot be satisfied. In order for SH surface waves to exist the solid half space must be covered with an extra surface layer with special properties, i.e., with a layer whose shear wave velocity is lower than that in bulk of the solid. This fact was demonstrated mathematically in 1911 by Augustus Edward Hough Love and as a consequence SH surface waves in a layered half space were named Love surface wave [2–4].

It is interesting to note that SH surface waves of the Love type have their direct counterparts in electromagnetic and optical planar waveguides [5–6], described by celebrated Maxwell's equations.

Other significant similarities exist between the properties of Love surface waves and behavior of quantum particles moving in a rectangular potential well [7]. This quantum problem is described by a time-independent Schrödinger equation with appropriate boundary conditions.

Due to the above analogies, occurring between Love surface waves, optical dielectric waveguides and quantum particles in a rectangular potential well, the developments made in one domain (elasticity, electromagnetism, optics or quantum mechanics) can be transferred to the remaining domains. These interdisciplinary similarities are not only very inspiring and useful in practical applications, but also they demonstrate clearly a unifying power of the underlying mathematical theory.

Shear horizontal (SH) surface acoustic waves of the Love type and bulk acoustic waves are in the core of many scientific and engineering domains, such as geophysics, seismology and earthquake engineering [8–13], nondestructive testing (NDT) and material characterization [14–19], viscosity sensors, [20–24], biosensors, signal processing [25], etc.

Classical surface waves of the Love type were initially analyzed in lossless waveguides, consisting of an elastic surface layer deposited on an elastic substrate. As a result, the wave number k_0 of the classical Love wave was real and the wave propagated unattenuated, i.e., its amplitude was constant as a function of the propagation distance. Classical surface waves of the Love type propagating in lossless waveguides, have closed form analytical solutions for their amplitude and a dispersion relation for the phase velocity in a form of a transcendental algebraic equation [2–4].

Very few papers were published up to date on Love surface waves propagating in lossy waveguides, i.e., in waveguides whose surface layer and/or substrate are lossy. First step in this direction was made by a celebrated seismologist Katsutada Sezawa in 1938 [26]. However, due to enormous mathematical difficulties he was not able to solve this problem, mainly due to the fact that his work was done before the era of digital computers. Similar problem was discussed in a number of subsequent publications [27–29]. However, all these papers are rather generic and do not provide specific solutions of the dispersion relation (phase velocity and attenuation) of Love surface waves propagating in layered lossy waveguides.

The examination of Love surface waves propagating in lossy waveguides is of primary importance, from the cognitive and practical point of view, in geophysics, seismology [30], nondestructive evaluation of materials, sensors [31–34], etc. Indeed, measurements of dispersion curves (phase velocity and attenuation) for Love surface waves propagating on the Earth's surface provide important information about physical properties of the Earth's crust. Similarly, knowledge of the phase velocity and attenuation for Love waves propagating in lossy waveguides is indispensable in modeling and design of bio and chemosensors [31], whose sensing surface layer is usually made from a lossy material. The use of Rayleigh surface waves in bio and chemosensors with a lossy surface layer has been considered in [35].

The fundamental goal of this paper is to formulate and solve the Direct Sturm Liouville problem that describes the propagation of Love waves in viscoelastic layered waveguides. Solution of the Direct Sturm–Liouville problem allows to evaluate the impact of losses in the surface layer on the propagation characteristics of the Love waves.

Motivated by incompleteness of analytical and/or numerical solutions the Author analyzes in this paper propagation of Love surface waves in lossy planar waveguides, consisting of a lossy viscoelastic surface layer deposited on a lossless elastic substrate. Since part of the energy of the Love surface wave will be inevitably converted to heat, in the lossy surface layer, the Love surface wave will be attenuated and therefore its wave number k will be described by a complex quantity, namely $k = k_0 + j\alpha$, where $k_0 = \omega/v_p$, ω is the angular frequency of the wave, v_p is its phase velocity and α is the coefficient of attenuation. By contrast to the lossless case the dispersion relation developed in this paper is complex, i.e., contains functions of complex valued quantities.

It is worth noticing that, the introduction of losses significantly complicates the analysis and solution of the problem of Love wave propagation in viscoelastic waveguides. Final calculations have to be performed numerically, using the appropriate numerical procedures.

To solve numerically the resulting complex dispersion relation its real and imaginary part were separated providing a set of two nonlinear algebraic equations for k_0 and α, what is a novelty. This set of nonlinear equations was then solved numerically. Phase velocity $v_p = \omega/k_0$ and coefficient of attenuation α were subsequently calculated as a function of the wave frequency f, thickness of the surface layer h and its viscosity η_{44}. Dispersion curves for Love waves propagating in lossy waveguides (with a lossy surface layer deposited on a lossless substrate) were compared to those corresponding to Love surface waves propagating in lossless waveguides, with a lossless surface layer deposited on a lossless substrate.

The results obtained in this paper are original and to some extent unexpected. For example, it was found that: 1) the phase velocity v_p of Love surface waves increases as a function of viscosity η_{44} of the lossy surface layer, and 2) the coefficient of attenuation α has a pronounced maximum as a function of thickness h of the lossy surface layer.

Surface waves of the Love type display a multimode structure. In theory, they have an infinite number of modes with different amplitude distributions, phase velocities and cut-off frequencies. However, in many practical applications, such as

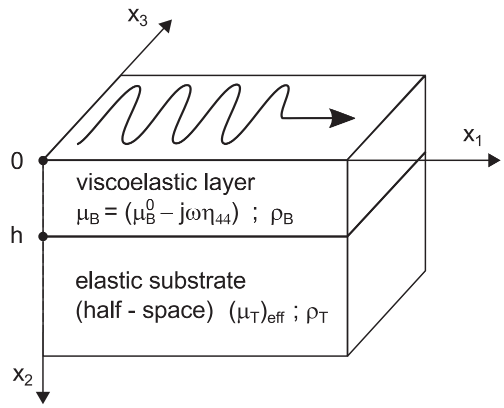

Fig. 1. Geometry of the SH Love surface wave waveguide with a lossy viscoelastic surface layer of thickness h, deposited on a lossless elastic half space. Love surface waves are polarized along the x_3 axis, they propagate in the x_1 direction and their amplitude decays asymptotically along the x_2 axis.

in sensors and NDT, most important is the fundamental mode with a zero cut-off frequency. Therefore, this paper restricts its scope only to the fundamental Love wave mode.

Presented in this paper formulation and solution of the Direct Sturm–Liouville problem, for Love waves propagating in viscoelastic waveguides, is a novelty.

The layout of this paper is organized as follows. Section 2 presents general form of the Love surface wave and parameters of the lossy waveguide with a lossy surface layer deposited on a lossless substrate. Losses in the surface layer are modeled by a frequency independent viscosity η_{44}, being part of the Kelvin–Voigt model of the viscoelastic material. Section 3 contains the mathematical model describing propagation of Love surface waves in the lossy waveguide, in terms of the direct Sturm–Liouville problem. In Section 4, we establish the complex dispersion relation for the phase velocity v_p and attenuation α of the Love wave. Section 5 shows the results of numerical calculations for the lossy waveguide with a poly(methyl methacrylate) (PMMA) surface layer deposited on a quartz substrate. Discussion and conclusions are given, respectively, in Sections 6 and 7.

2. Material parameters and geometry of the Love wave surface waveguide

The geometry of the waveguide structure with a lossy (viscoelastic) surface layer of thickness h, deposited on a lossless elastic half space is shown in Fig. 1. The mechanical displacement u_3 of the SH surface wave of the Love type is directed along the x_3 axis, which is perpendicular to the direction of propagation x_1 and perpendicular to the x_2 axis pointing into the bulk of the substrate. Love wave is a surface wave, hence the mechanical displacement u_3 of the Love wave should vanish in the bulk of the substrate for $x_2 \to \infty$. The thickness of the lossy (viscoelastic) surface layer is h. It was assumed that the amplitude u_3 of the Love surface wave is constant along the x_3 axis and the wavefronts are infinitely extended along the x_3 direction. As a result, all partial derivatives $\partial/\partial x_3$ over the axis x_3 vanish.

In this paper it is assumed that the analyzed surface wave of the Love type is time harmonic, i.e., its propagation is described by an exponential propagation factor $exp[j(k-\omega t)]$, where $j=\sqrt{-1}$, k is the wave number of the Love wave and ω its angular frequency. Since surface layer is lossy (viscoelastic) the wave number k of the Love wave will be in general a complex-value quantity:

$$k = k_0 + j\alpha \tag{1}$$

where the real part k_0 of the wave number determines phase velocity $v_p = \omega/k_0$ of the Love wave and the imaginary part α of the wave number is an attenuation coefficient of the Love wave.

Classical theory of elasticity assumes that during deformation the material can store elastic energy without losses (dissipation). However, all real materials display some sort of lossy behavior. This is especially true in polymers, composites, liquid crystals or lubricants. The behavior of these materials combines energy storing properties of elastic solids with energy dissipation properties of viscous liquids. Therefore, such materials were named viscoelastic materials. For time-harmonic waves $e^{-j\omega t}$ elastic moduli of viscoelastic materials are complex and frequency dependent. The real part of the elastic modulus represents the capacity of the material to store elastic energy, whereas the imaginary part represents the loss of energy, converted irreversibly to heat.

In this paper viscoelastic properties of the lossy surface layer are described by the Kelvin–Voigt viscoelastic medium [36]. The shear modulus μ_B of the surface layer is a complex quantity and can be represented as follows:

$$\mu_B = \mu_B^0 - j\omega\eta_{44} \tag{2}$$

where μ_B^0 is the storage shear modulus of the viscoelastic surface layer, η_{44} is the viscosity of the viscoelastic layer, ω is an angular frequency and $j = \sqrt{-1}$. By contrast, the shear modulus μ_T of the lossless elastic substrate is a real number and does not depend on the wave frequency.

Mechanical losses in viscoelastic materials, for shear vibrations, can be totally described by one parameter, i.e., by the loss tangent $tan\delta$ of the material, where δ is the phase angle between the shear stress and shear strain in the material. The inverse of the loss tangent $tan\delta$ is equal to the mechanical quality factor Q of the material. For bulk SH shear waves, in the viscoelastic materials, the loss tangent is given by: $1/Q = tan\delta = \omega\eta_{44}/\mu_B^0$. Typical values of the loss tangent for SH waves range from 10^{-5} in single-crystal materials to $10^{-3} - 10^{-1}$ in polycrystalline solids and solid polymers [37]. On the other hand, the mechanical quality factor Q for the SH waves propagating in the upper Earth crust, is of the order of $Q = 50$ [38]. Thus, the corresponding loss tangent $tan\delta$ is equal to $tan\delta = 2 \times 10^{-2}$, a value much lower than 1.

It is assumed throughout this paper that all materials of the layered waveguide structure shown in Fig. 1, i.e., lossy (viscoelastic) surface layer and lossless elastic substrate are linear, isotropic (only for surface layer), homogeneous, frequency independent and time-invariant. In addition, the mechanical losses in the viscoelastic surface layer are proportional to the constant shear viscosity η_{44}. The viscoelastic properties of the lossy surface layer are adequately described by the Kelvin–Voigt model valid for solid-like viscoelastic materials.

3. Mathematical formulation of the problem

Love wave propagation in layered viscoelastic media can be described in terms of the Direct Sturm–Liouville problem. Determination of the phase velocity and attenuation (eigenvalues), and mechanical displacement distribution with depth (eigenvectors) of the SH surface Love wave, from a knowledge of elastic and viscoelastic parameters of a layered half-space, constitutes a direct Sturm–Liouville problem.

Mathematical analysis of Love surface waves propagating in the surface waveguide with a lossy viscoelastic surface layer $0 \le x_2 < h$ deposited on a lossless elastic substrate $x_2 \ge h$ begins with Newton's equations of motion (partial differential equations of the second order) written separately for the lossy surface layer and lossless substrate. Subsequently, the appropriate boundary conditions are formulated for the shear stress τ_{23} and shear mechanical displacement u_3 of the Love wave.

3.1. Governing differential equations

In the following of this paper we will analyze properties of SH surface waves of the Love type propagating in lossy waveguides composed of an isotropic viscoelastic (lossy) layer of thickness h, with the complex shear modulus $\mu_B = \mu_B^0 - j\omega\eta_{44}$, and density ρ_B, overlying a lossless elastic substrate, with an effective shear modulus of elasticity $(\mu_T)_{eff}$, see Fig. 1.

3.1.1. Viscoelastic surface layer ($0 \le x_2 < h$)

The mechanical displacement field $u_3^{(1)}$ of the SH Love wave in the viscoelastic surface layer ($0 \le x_2 < h$) is governed by the following equation of motion:

$$\frac{1}{v_1^2}\frac{\partial^2 u_3^{(1)}}{\partial t^2} = \left(\frac{\partial^2}{\partial \boldsymbol{x}_1^2} + \frac{\partial^2}{\partial x_2^2}\right)u_3^{(1)} \tag{3}$$

where $v_1 = ((\mu_B^0 - j\omega\eta_{44})/\rho_B)^{1/2} = v_1^0(1 - j\frac{\omega\eta_{44}}{\mu_B^0})^{1/2}$ is the complex bulk shear wave velocity in the layer, $v_1^0 = (\mu_B^0/\rho_B)^{1/2}$ is the bulk shear wave velocity in a corresponding lossless surface layer, and ρ_B is the density of the surface layer.

3.1.2. Elastic substrate ($x_2 \ge h$)

The mechanical displacement field $u_3^{(2)}$ of the SH Love wave in the lossless elastic substrate ($x_2 \ge h$) satisfies the following equation of motion:

$$\frac{1}{v_2^2}\frac{\partial^2 u_3^{(2)}}{\partial t^2} = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)u_3^{(2)} \tag{4}$$

where: $v_2 = ((\mu_T)_{eff}/\rho_T)^{1/2}$ is the phase velocity of bulk shear waves in the substrate, $(\mu_T)_{eff}$ is the effective shear modulus of elasticity in the elastic substrate, and ρ_T is the density of the elastic substrate.

3.2. Boundary conditions

The mechanical displacement u_3 of the surface Love wave, in the substrate ($x_2 \ge h$), should vanish for large distances from the interface ($x_2 = h$), i.e., for $x_2 \to \infty$. At the interface ($x_2 = h$) between the viscoelastic surface layer and the substrate, the conditions of continuity for the mechanical displacement and shear stress must be satisfied. Moreover, at the free surface ($x_2 = 0$) the shear stress is equal to zero. Therefore, the boundary conditions are as follows:

1. On a free surface ($x_2 = 0$), the transverse shear stress $\tau_{23}^{(1)}$ is equal to zero, hence:

$$\tau_{23}^{(1)}\Big|_{x_2=0} = \mu_B \frac{\partial u_3^{(1)}}{\partial x_2}\Bigg|_{x_2=0} = 0 \tag{5}$$

2. The continuity of the displacement field u_3 and stress τ_{23} at the interface between the viscoelastic layer and the substrate ($x_2 = h$) gives rise to:

$$u_3^{(1)}\Big|_{x_2=h} = u_3^{(2)}\Big|_{x_2=h} \tag{6}$$

$$\tau_{23}^{(1)}\Big|_{x_2=h} = \mu_B \frac{\partial u_3^{(1)}}{\partial x_2}\Bigg|_{x_2=h} = \tau_{23}^{(2)}\Big|_{x_2=h} = (\mu_T)_{eff} \frac{\partial u_3^{(1)}}{\partial x_2}\Bigg|_{x_2=h} \tag{7}$$

3. $u_3 \to 0$ when $x_2 \to \infty$.

3.3. *Direct Sturm–Liouville problem*

Mechanical displacement $u_3 = [u_3^{(1)},\ u_3^{(2)}]^T$ of a time-harmonic Love surface wave, propagating along the axis x_1, and polarized along the x_3 axis, will be sought in the following form:

$$u_3^{(1)}(x_1, x_2, t) = f_1(x_2) exp[j(k \cdot x_1 - \omega t)] \quad ; \quad (0 \le x_2 < h) \tag{8}$$

$$u_3^{(2)}(x_1, x_2, t) = f_2(x_2) exp[j(k \cdot x_1 - \omega t)] \quad ; \quad (x_2 \ge h) \tag{9}$$

where: the function $f(x_2) = [f_1(x_2), f_2(x_2)]^T$ describes variations of the mechanical displacement u_3 as a function of depth (x_2 direction), and k is the wave number of the Love wave, given by Eq. (1). Here: $f_1(x_2)$ is the distribution with the depth of the mechanical displacement in the layer, and $f_2(x_2)$ is the distribution with the depth of the mechanical displacement in the substrate.

After substitution of Eqs. (8) and (9) into equations of motion (Eqs. (3) and (4)), one obtains:

$$f_1''(x_2) + k_1^2 \cdot f_1(x_2) = k^2 \cdot f_1(x_2) \quad ; \quad (0 \le x_2 < h) \tag{10}$$

$$f_2''(x_2) + k_1^2 \cdot f_2(x_2) = k^2 \cdot f_2(x_2) \quad ; \quad (x_2 \ge h) \tag{11}$$

where $k_1 = \omega / v_1$; $k_2 = \omega / v_2$, $v_1 = ((\mu_B^0 - j\omega\eta_{44})/\rho_B)^{1/2}$, and $v_2 = ((\mu_T)_{eff}/\rho_T)^{1/2}$.

The superscript "prime" denotes the differentiation with respect to the variable x_2. It is noteworthy that the phase velocity v_1 in the surface layer is a complex quantity.

Substitution of Eqs. (8) and (9) into boundary conditions (Eqs. (5)–(7)) results in:

$$f_1'(x_2)\Big|_{x_2=0} = 0 \tag{12}$$

$$f_1(x_2)|_{x_2=h} = f_2(x_2)|_{x_2=h} \tag{13}$$

$$\mu_B\ f'_1(x_2)\Big|_{x_2=h} = (\mu_T)_{eff}\ f'_2(x_2)\Big|_{x_2=h} \tag{14}$$

The differential problem (10)–(11), with boundary conditions (12)–(14), which consists in determining the pairs $\{f(x_2), k^2\}$ that satisfy the Eqs. (10)–(11) and the boundary conditions (12)–(14), forms a Direct Sturm–Liouville problem, wherein: $f(x_2) = [f_1(x_2), f_2(x_2)]^T$ is the eigenvector, and k^2 is the eigenvalue. Both, the eigenvector and the eigenvalue are complex quantities in lossy waveguides.

3.4. *Propagating wave solution*

3.4.1. *Viscoelastic surface layer* ($0 \le x_2 < h$)

The solution of Eq. (10) can be expressed as:

$$f_1(x_2) = C_1 \cdot \sin(q_B \cdot x_2) + C_2 \cdot \cos(q_B \cdot x_2) \tag{15}$$

where:

$$q_B = \left(k_1^2 - k^2\right)^{1/2} \quad ; \quad k_1 = \frac{\omega}{v_1}$$

C_1 and C_2 are arbitrary constants

3.4.2. Elastic substrate ($x_2 \geq h$)

Since the amplitude of Love surface waves for $x_2 \to \infty$ vanishes, the solution to Eq. (11) is sought as:

$$f_2(x_2) = C_3 \cdot exp(-b \cdot x_2) \tag{16}$$

where $b = (k^2 - k_2^2)^{1/2}$

$$k_2 = \frac{\omega}{v_2} \text{ and } Re(b) > 0,$$

C_3 is an arbitrary constant.

This solution represents SH surface wave that amplitude decays to zero, when $x_2 \to \infty$.

4. Complex dispersion equation

After substitution of Eqs. (15) and (16) into boundary conditions (12–14), the set of three linear and homogeneous equations for coefficients C_1, C_2, and C_3 is obtained. For nontrivial solution, the determinant of this set of linear algebraic equations has to equal zero (necessary condition). This leads to the following complex dispersion relation:

$$\sin(q_B \cdot h) \cdot \mu_B \cdot q_B - \cos(q_B \cdot h) \cdot (\mu_T)_{eff} \cdot b = 0 \tag{17}$$

In the dispersion Eq. (17) the quantities μ_B, b *and* q_B are complex, namely:

$$q_B = \sqrt{\left(K_1^2 \frac{1}{\left(1 + tan^2\delta\right)} - k_0^2 + \alpha^2\right) + j \cdot \left(K_1^2 \frac{tan\delta}{\left(1 + tan^2\delta\right)} - 2 \cdot k_0 \cdot \alpha\right)} \tag{18}$$

$$b = \sqrt{\left(k_0^2 - \alpha^2 - k_2^2\right) + j \cdot 2 \cdot k_0 \cdot \alpha} \tag{19}$$

$$\mu_B = \mu_B^0 - j\omega\eta_{44} = \mu_B^0\left(1 - jtan\delta\right) \tag{20}$$

where $K_1 = \omega / v_1^0$; $k_2 = \frac{\omega}{v_2}$; $k_0 = \frac{\omega}{v_p}$ and $tan\delta = (\frac{\omega\eta_{44}}{\mu_B^0})$ are real variables.

Eq. (17) is the complex dispersion equation of Love waves propagating in lossy layered waveguides composed of a viscoelastic surface guiding layer attached to a lossless elastic substrate (half-space). The dispersion Eq. (17) can be considered to be an implicit equation for the phase velocity v_p and the attenuation α of the Love surface wave, as a function of the angular frequency ω, the thickness h of the surface layer and other relevant material parameters of the lossy surface layer and the lossless substrate.

In this paper, numerical calculations are performed for Love surface waves propagating in the layered waveguides composed of a poly(methyl methacrylate) (PMMA) surface layer deposited on lossless quartz substrate. In this case, for typical values of the frequency that are used in sensor technology (of the order of several MHz) and viscosity η_{44} (up to 50 *Pas*) the values of k_0 and α can be estimated as follows: $k_0 < 2\pi$ x 3000 and $\alpha < 2000$. For that set of parameters, second terms under the square root in Eqs. (18) and (19) are smaller than the corresponding first terms. Similar estimation is also valid for high frequency (100–300 MHz) waveguides with surface layers of lower viscosities of the order of 1 mPas. Therefore, it will be convenient to simplify the Eqs. (18) and (19) by expanding them in a complex Taylor series and retaining only first order approximation terms: $F(z) = \sqrt{1+z} \approx 1 + z/2$. Here: $F(z)$ represents symbolically Eqs. (18) and (19) treated as functions of a complex variable z. This expansion converges for $|z| < 1$ [39].

Consequently, employing this Taylor expansion we can write:

$$q_B = \sqrt{\left(K_1^2 \frac{1}{\left(1 + tan^2\delta\right)} - k_0^2 + \alpha^2\right)} + j \frac{\left(\frac{1}{2} K_1^2 \frac{tan\delta}{(1+tan^2\delta)} - k_0 \cdot \alpha\right)}{\sqrt{\left(K_1^2 \frac{1}{(1+tan^2\delta)} - k_0^2 + \alpha^2\right)}} = c + jd \tag{21}$$

where $c = \sqrt{\left(K_1^2 \frac{1}{(1+tan^2\delta)} - k_0^2 + \alpha^2\right)}$, $d = \frac{\left(\frac{1}{2} K_1^2 \frac{tan\delta}{(1+tan^2\delta)} - k_0 \cdot \alpha\right)}{\sqrt{\left(K_1^2 \frac{1}{(1+tan^2\delta)} - k_0^2 + \alpha^2\right)}}$.

$$b = \sqrt{\left(k_0^2 - \alpha^2 - k_2^2\right)} + j \frac{k_0 \cdot \alpha}{\sqrt{\left(k_0^2 - \alpha^2 - k_2^2\right)}} \tag{22}$$

Substituting Eqs. (21) and (22) to the dispersion Eq. (17), and grouping the real and imaginary terms we obtain the following set of two nonlinear algebraic equations comprising only real variables:

$$\left(\sqrt{\left(K_1^2\frac{1}{(1+tan^2\delta)}-k_0^2+\alpha^2\right)}+\frac{\left(\frac{1}{2}K_1^2\frac{tan\delta}{(1+tan^2\delta)}\;-\;k_0\cdot\alpha\right)\cdot tan\delta}{\sqrt{\left(K_1^2\frac{1}{(1+tan^2\delta)}\;-\;k_0^2+\alpha^2\right)}}\right)-\cot(c\cdot h)\cdot\tanh(d\cdot h)\cdot$$
$$\left(\frac{\left(\frac{1}{2}K_1^2\frac{tan\delta}{(1+tan^2\delta)}\;-\;k_0\cdot\alpha\right)}{\sqrt{\left(K_1^2\frac{1}{(1+tan^2\delta)}\;-\;k_0^2+\alpha^2\right)}}-\sqrt{\left(K_1^2\frac{1}{(1+tan^2\delta)}-k_0^2+\alpha^2\right)}\cdot tan\delta\right)$$
$$-\frac{(\mu_T)_{eff}}{\mu_B^0}\cdot\left(\cot(c\cdot h)\cdot\sqrt{\left(k_0^2-\alpha^2-k_2^2\right)}+\frac{k_0\cdot\alpha}{\sqrt{\left(k_0^2-\alpha^2-k_2^2\right)}}\cdot\tanh(d\cdot h)\right)=0 \quad (23)$$

and:

$$\cot(c\cdot h)\cdot\tanh(d\cdot h)\cdot\left(\sqrt{\left(K_1^2\frac{1}{(1+tan^2\delta)}-k_0^2+\alpha^2\right)}+\frac{\left(\frac{1}{2}K_1^2\frac{tan\delta}{(1+tan^2\delta)}\;-\;k_0\cdot\alpha\right)\cdot tan\delta}{\sqrt{\left(K_1^2\frac{1}{(1+tan^2\delta)}\;-\;k_0^2+\alpha^2\right)}}\right)$$
$$+\left(\frac{\left(\frac{1}{2}K_1^2\frac{tan\delta}{(1+tan^2\delta)}\;-\;k_0\cdot\alpha\right)}{\sqrt{\left(K_1^2\frac{1}{(1+tan^2\delta)}\;-\;k_0^2+\alpha^2\right)}}-\sqrt{\left(K_1^2\frac{1}{(1+tan^2\delta)}-k_0^2+\alpha^2\right)}\cdot tan\delta\right)$$
$$-\frac{(\mu_T)_{eff}}{\mu_B^0}\cdot\left(\cot(c\cdot h)\cdot\frac{k_0\cdot\alpha}{\sqrt{\left(k_0^2-\alpha^2-k_2^2\right)}}-\sqrt{\left(k_0^2-\alpha^2-k_2^2\right)}\cdot\tanh(d\cdot h)\right)=0 \quad (24)$$

The formulas 23 and 24 can be written in a more abstract form as:

$$A\left(\mu_B^0,\ \rho_B,\ (\mu_T)_{eff},\ \rho_T,\eta_{44},\ h,\ \omega;\ k_0,\ \alpha\ \right)=0 \quad (25)$$

$$B\left(\mu_B^0,\ \rho_B,\ (\mu_T)_{eff},\ \rho_T,\ \eta_{44},\ h,\ \omega;\ k_0,\ \alpha\ \right)=0 \quad (26)$$

This is a system of two nonlinear algebraic equations. The unknowns are: k_0 and α.

The parameters are: $\mu_B, \rho_B, (\mu_T)_{eff}, \rho_T, \eta_{44}, h$ *and* ω.

Eqs. (23) and (24) contain only real variables (unknowns and parameters) and therefore they can be regarded as a real equivalent of the complex dispersion relation Eq. (17), describing properties of Love surface waves propagating in layered lossy waveguides composed of a viscoelastic surface layer deposited on an elastic lossless substrate (half-space).

The system of nonlinear Eqs. (23) and (24) was solved using the numerical Powell hybrid method implemented in a computer package Scilab. This method for the efficient solution of Eqs. (23) and (24) requires an appropriate choice of initial approximations. After finding a solution (k_0, α), one can calculate the phase velocity of the Love wave $v_p = \omega/k_0$. Imaginary part α of the wavenumber k represents the attenuation of the Love wave per unit length in the direction of propagation.

5. Results of numerical calculations

Numerical calculations were performed for the selected waveguide structure composed of a poly(methyl methacrylate) (PMMA) polymer surface layer deposited on a quartz substrate. Quartz has very low losses (up to frequencies 1 GHz), therefore it can be considered as a very good approximation for a lossless elastic substrate. The only source of losses in the waveguide is the viscosity of the polymeric surface layer.

In this paper, an ST- 90° X quartz plate was chosen as a substrate material for Love wave waveguides. It is a quartz crystal plate with ST-cut and wave propagation direction orthogonal to the X-axis. The quartz plate with this cut and wave propagation direction is characterized by the following Euler angles (0°, 132.75°, 90°). The bulk shear wave propagates along a free surface of the plate in the direction of the Z' axis (tilted by an angle of 42.75° relative to the original Z crystallographic axis). The mechanical displacement of the shear ultrasonic wave is polarized along the X crystallographic axis, perpendicular to the direction of propagation Z'. The presented analysis shows that the pure transverse ultrasonic waves can propagate in the quartz medium specified above.

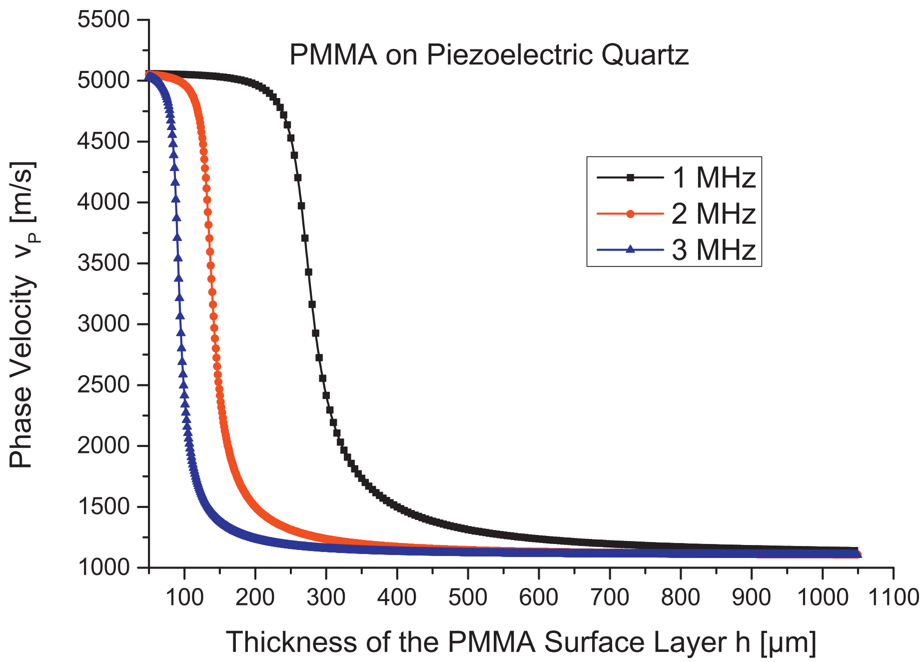

Fig. 2. Phase velocity of the Love surface wave versus thickness h of the surface layer for different frequencies and constant surface layer viscosity $\eta_{44} = 37 \times 10^{-2}$ Pas.

Anisotropic properties of quartz are taken into account by introducing the concept of effective shear modulus $(\mu_T)_{eff}$ which determines the velocity of the bulk shear wave.

In the mathematical analysis of propagation of Love waves, in the considered waveguide structure (PMMA layer on the quartz substrate), the elastic properties of the quartz semi-space are represented by (as in the isotropic body) two parameters i.e., the effective shear modulus $(\mu_T)_{eff}$ and the density ρ_T.

Piezoelectrically stiffened effective shear modulus $(\mu_T)_{eff}$ for shear bulk waves propagating in the direction of Z' axis with the polarization direction along the X axis is given by the following formula [40]:

$$(\mu_T)_{eff} = c_{66}sin^2\theta + c_{44}cos^2\theta - c_{14}sin2\theta + \frac{\left(e_{11}sin^2\theta + \frac{e_{14}}{2}\sin 2\theta\right)^2}{\epsilon_{11}sin^2\theta + \epsilon_{33}cos^2\theta} \tag{27}$$

where (c_{66}, c_{44}, c_{14}), (e_{11}, e_{14}) and $(\varepsilon_{11}, \varepsilon_{33})$ are elastic constants, piezoelectric coefficients and dielectric constants of the Quartz. The angle θ is the angle between the new rotated Z' axis and the original Z crystallographic axis in Quartz material. For ST-cut Quartz samples, the angle $\theta = +42.75^o$.

In the numerical computations the following values of material parameters in the waveguide were assumed:

For PMMA poly(methyl methacrylate)	For quartz
$\mu_B^0 = 1.43 \times 10^9$ N/m^2	$(\mu_T)_{eff} = 67.85 \times 10^9$ N/m^2
$\rho_B = 1.18 \times 10^3$ kg/m^3	$\rho_T = 2.56 \times 10^3$ kg/m^3
$v_1^0 = \left(\mu_B^0/\rho_B\right)^{1/2} = 1100$ m/s	$v_2 = \left((\mu_T)_{eff}/\rho_T\right)^{1/2} = 5060$ m/s

To illustrate the influence of viscoelastic parameters of the surface layer on the properties of Love waves, the numerical calculations were performed for ultrasonic frequencies ranging from 1 to 5 MHz and for the loss tangent ($tan\delta$) of the polymeric material varying from 0 to 0.8. The thickness h of the polymeric surface layer varies from 20 to 1200 μm.

In Section 5.4, we inserted additionally dispersion curves of Love waves that propagate in the waveguides with dimensions of the surface layer typical for bio and chemo microsensors (the thickness of the surface layer is of the order of 1 μm, and frequencies are above 100 MHz).

5.1. Phase velocity and attenuation versus thickness h *of the surface layer, (η_{44} = const)*

Fig. 2 shows the plot of the phase velocity v_p of the Love surface wave, as a function of the thickness h of the surface layer, for various frequencies of the surface wave, namely $f = 1,\ 2$ *and* 3 MHz.

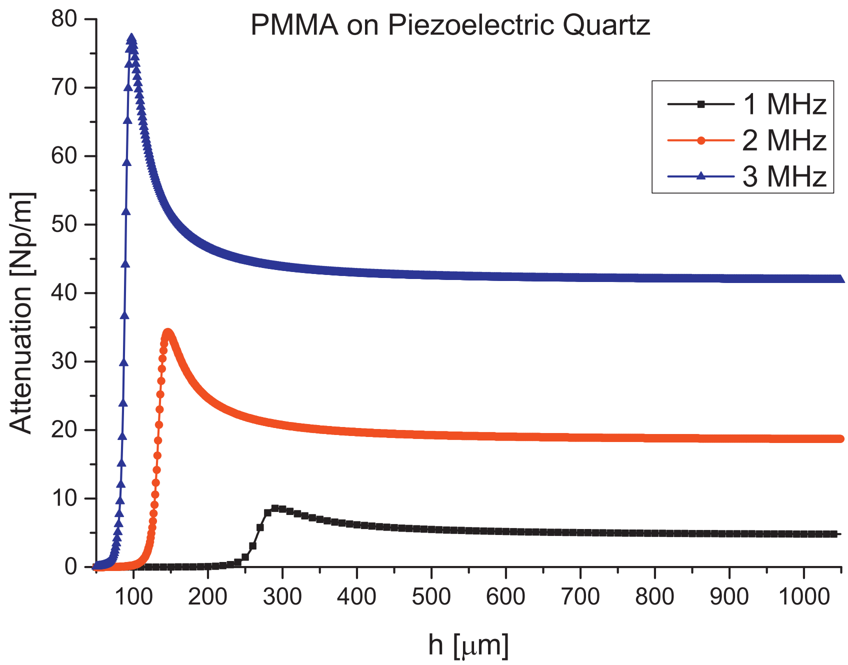

Fig. 3. Attenuation of the Love surface wave versus thickness h of the surface layer for different frequencies and constant surface layer viscosity $\eta_{44} = 37 \times 10^{-2}$ Pas.

As it is seen in Fig. 2 the phase velocity v_p of the Love surface wave begins at $v_2 = ((\mu_T)_{eff}/\rho_T)^{1/2} = 5060$ m/s, for $h = 0$ and decreases monotonically to the bulk shear velocity in the layer $v_1^0 = (\mu_B^0/\rho_B)^{1/2} = 1100$ m/s, for $h \to 0$. In other words, for thicker surface layer, properties of the Love surface wave are more influenced by properties of the layer.

Fig. 3 illustrates the dependence of the Love wave attenuation on layer thickness h.

Fig. 3 shows to some extent unexpectedly, that the attenuation α has a pronounced maximum as a function of thickness h. For increasing frequencies f of the Love surface wave the maximum occurs for lower thicknesses of the surface layer, i.e., for frequencies $f = 1,\ 2$ *and* 3 MHz the maximum occurs, respectively, for $h = 286,\ 144$ *and* 98 μm.

In paper [34], the maximum of attenuation of Love waves as a function of the thickness h of the surface layer was also observed. However, the source of attenuation observed in this article is mainly the viscosity of the viscous liquid which loads the surface of the layered waveguide. In the present paper, Love wave attenuation is due only to the viscosity of the viscoelastic surface layer.

5.2. Phase velocity versus frequency, (h = const)

The dependence of the phase velocity on the frequency for different values of the loss tangent is shown in Fig. 4.

As can be seen in Fig. 4, The phase velocity depends slightly on the loss tangent $tan\delta$. With the increase of losses the phase velocity v_p of the Love wave slightly increases. In general, the influence of the frequency on the phase velocity is similar to the influence of the thickness h (see Fig. 2).

5.3. Phase velocity and attenuation versus loss tangent for various values of layer thickness h, (f = const)

Figs. 5 and 6 display the dependence of the phase velocity v_p and attenuation α on the loss tangent of the surface layer.

It is seen from Figs. 5 and 6 that the phase velocity and attenuation follow the same trend, namely they increase monotonically with the increasing loss tangent of the surface layer. It is to some extent surprising, but the phase velocity v_p of the Love surface wave is a growing function of the losses in the surface layer.

5.4. Dispersion curves of the Love wave propagating in the waveguide structures typical for bio and chemo microsensors

In this subsection, examples of the dispersion curves for Love waves, that propagate in the waveguides structures typical for bio and chemosensors, will be given. In these sensors the frequencies of the Love wave above 100 MHz and thicknesses h of the surface layer of the order of 1 μm are employed.

In numerical calculations, the elastic parameters of the quartz substrate and surface layer (PMMA) were used the same as in Section 5.1. Since the attenuation of the Love wave grows approximately as a square of the frequency, therefore for frequencies above 100 MHz, the PMMA viscosity value has been reduced to 1 mPas.

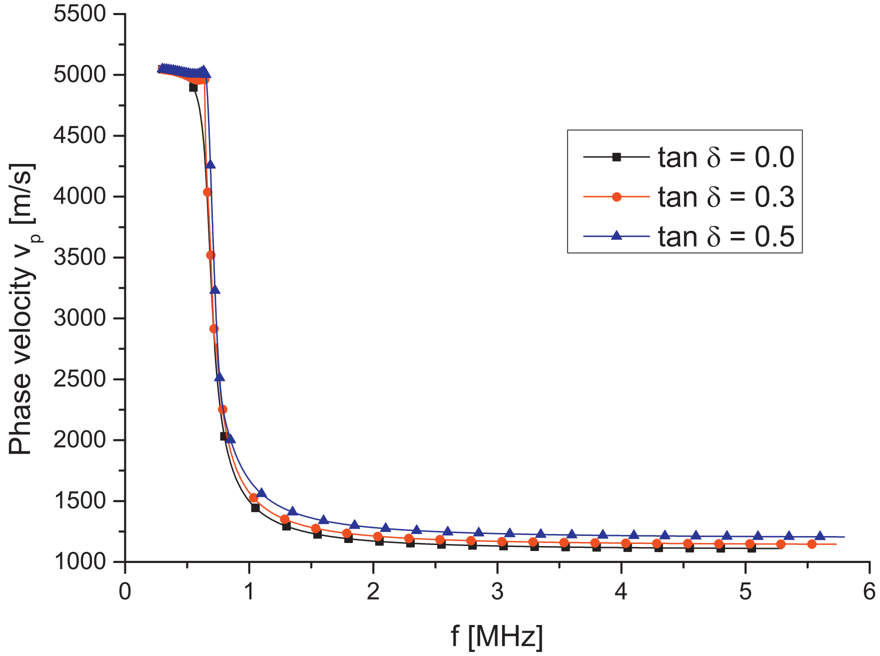

Fig. 4. Phase velocity of the Love surface wave versus frequency for different loss tangent in the surface layer and constant surface layer thickness $h = 400$ μm.

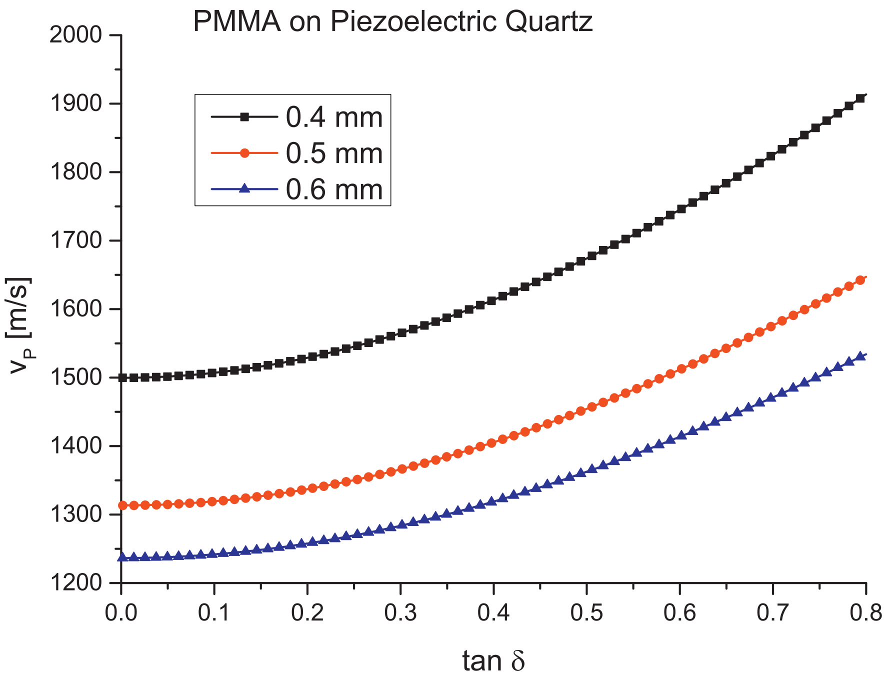

Fig. 5. Phase velocity of the Love surface wave versus loss tangent of the surface layer for different thicknesses of the surface layer and constant frequency $f = 1$ MHz.

Fig. 7 represents the dependence of the phase velocity of the Love wave, that propagates in the layered waveguide structure, on the thickness *h* of the viscoelastic surface layer.

As one can notice in Fig. 7, the phase velocity of the Love wave decreases monotonically with the increase of the surface layer thickness *h*.

The dependence of the Love wave attenuation versus thickness *h* of the surface layer is given in Fig. 8, for frequency as a parameter.

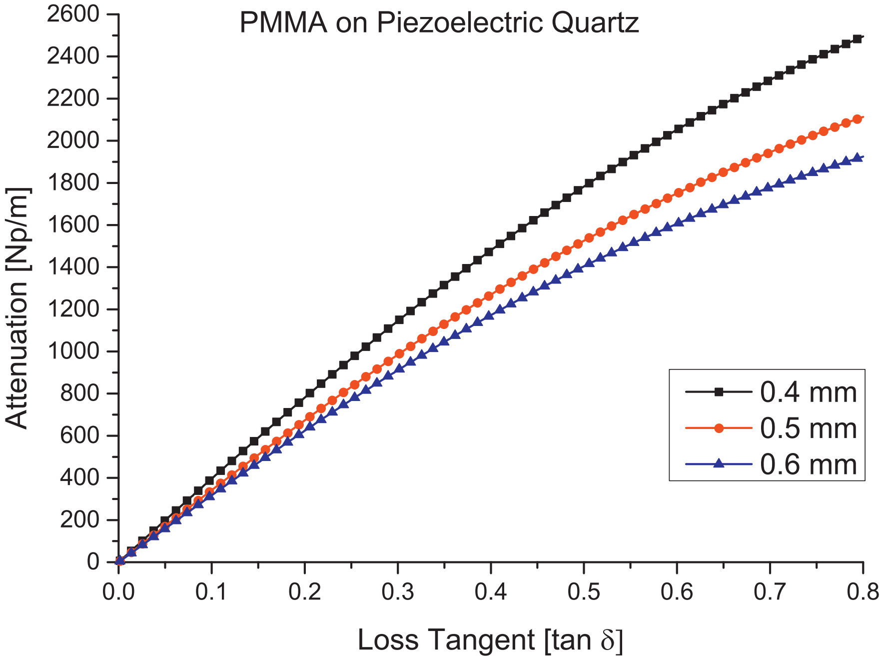

Fig. 6. Attenuation of the Love surface wave versus loss tangent of the surface layer for different thicknesses of the surface layer and constant frequency $f = 1$ MHz.

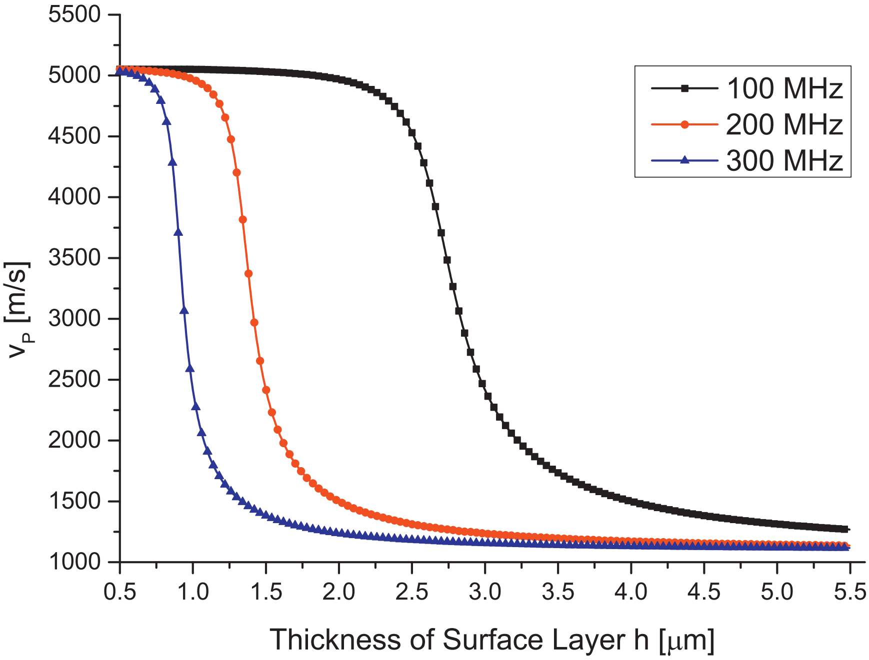

Fig. 7. Phase velocity of the Love surface wave versus thickness h of the surface layer for different values of frequency and constant surface layer viscosity $\eta_{44} = 1 \times 10^{-3}$ Pa s.

Fig. 8 displays an interesting feature, i.e., the attenuation of the Love wave exhibits maximum as a function of the surface layer thickness h.

6. Discussion

An novel effect observed in Figs. 3 and 8 shows that the attenuation α of the Love surface wave is not a monotonic function of the surface layer thickness h. Indeed, Figs. 3 and 8 show that for a given frequency f, the attenuation α of the

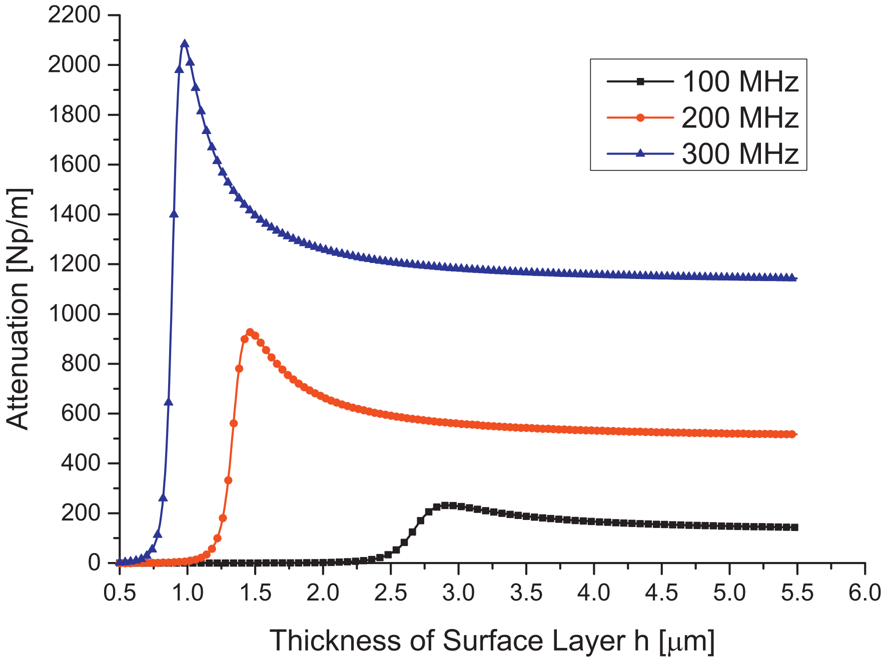

Fig. 8. Attenuation of the Love surface wave versus thickness *h* of the surface layer for different frequencies and constant surface layer viscosity $\eta_{44} = 1 \times 10^{-3}$ Pa s.

Love surface wave attains a maximum, as a function of the thickness *h* of the surface layer. The maximum attenuation for the frequency $f = 2$ MHz is exactly 4 times larger than that for $f = 1$ MHz. Similarly, the maximum attenuation for $f = 3$ MHz is 9 times greater than that for $f = 1$ MHz. Thus, the attenuation of the Love surface wave (at the maximum points) increases exactly as a square of the frequency ($\alpha \sim f^2$). It is worth noting that, this dependence ($\alpha \sim f^2$) applies exactly to the acoustic bulk shear waves propagating in viscoelastic media [41]. Figs. 3 and 8 show that the square dependence of the attenuation of the Love surface wave, on the frequency ($\alpha \sim f^2$), is also approximately satisfied outside the maximum points for another values of the thickness *h*.

Introduction of losses into the surface layer, in addition to the obvious effect which gives rise to the attenuation of the Love wave, also results in a less evident effect which is the increase of the Love wave phase velocity. This phenomenon, already noticed for bulk waves in [42], can be explained by the increase of an apparent shear stiffness of the material that results from the presence of additional viscous forces due to the viscosity of the surface layer. From Figs. 4, 5 it is evident that, phase velocity of the Love wave increases with the increasing losses in the surface layer. A similar phenomenon (rise in the phase velocity with losses) was also observed for Rayleigh surface waves propagating in viscoelastic media (half-spaces) [43–45].

It has already been proven that for low values of losses, attenuation of the Love wave depends linearly on the loss tangent [19]. In the present work, we have extended this analysis to larger values of losses, i.e., to $tan\delta = 0.8$. From the numerical calculations carried out by the Author, see Fig. 6, it follows that attenuation of the Love wave is a linear function of the loss tangent up to approximately $tan\delta = 0.1$. For higher values of the loss tangent attenuation curves of the Love wave saturate.

It is worth noting that the dispersion curves of Love waves that propagate in layered waveguide structures with the following parameters: 1) the thickness of the surface layer is of the order of micrometers (frequency range above 100 MHz), and 2) the thickness of the surface layer is of the order of a fraction of a millimeter (frequency range above 1 MHz) are qualitatively similar. This can be explained by the fact that the attenuation in the surface layer material increases as a square of the wave frequency. Thus, high frequency Love waves are sensitive to very small changes in viscosity of the surface layer and are used in microsensors employed in biology and chemistry.

7. Conclusions

In this study, the Direct Sturm–Liouville problem describing the propagation of Love surface waves in viscoelastic waveguides was formulated and solved, what is a novelty.

Theoretical analysis and numerical calculations were was performed for Love surface waves propagating in lossy waveguides consisting of a viscoelastic guiding surface layer deposited on a lossless elastic substrate (half-space).

The complex dispersion equation has been derived. The real and imaginary parts of this complex equation were separated, what is an original procedure that facilitates the solution of the complex dispersion equation. The resulting set of two nonlinear algebraic equations was then solved numerically, enabling the determination of the dispersion curves of phase velocity v_p and attenuation α of Love waves.

Established in this paper analytical formulas in the form of a system of nonlinear algebraic equations, allow to estimate the impact of various material parameters on the propagation of Love waves. These formulas can be used in the design and optimization of the viscosity sensors based on the use of Love waves, e.g., to evaluate the sensitivity of the sensor. By contrast, direct numerical solution of the complex dispersion equation cannot provide these evaluations.

An important contribution of this paper is the analysis of the impact of mechanical losses on the dispersion curves of the phase velocity and attenuation of Love waves.

In this work, the impact of surface layer viscosity η_{44} on the Love wave dispersion curves has been investigated. From the performed analysis and numerical calculations, one can conclude that:

(1) The increase in the viscosity η_{44} of the surface layer (and consequently also an increase in the loss tangent) results in an increase of the phase velocity of the Love wave, see Fig. 4. This is due to the viscosity of the layer, that causes an apparent stiffening of the material in the layer and an increase in the effective shear stiffness coefficient. From numerical calculations it follows that the phase velocity of the Love wave can significantly increase, even by 30%, e.g., for the thickness $h = 0.4$ mm, the frequency $f = 1$ MHz and $\tan\delta = 0.8$, the phase velocity of the Love wave increases from 1499.7 m/s to 1912 m/s, see Fig. 5.
(2) Dispersion curves for the attenuation α of the Love wave exhibit a maximum as a function of the thickness h of the surface layer. For a given frequency, there exists a thickness for which the attenuation factor α attains the maximum, see Fig. 3.
(3) The attenuation α of the Love wave depends approximately linearly on the tangent loss $\tan\delta$ of the surface layer and saturates for growing losses, see Fig. 6.

These characteristics of the Love wave dispersion curves in viscoelastic media are original and can be particularly useful in the optimum design and development of biosensors and chemosensors based on the use of surface waves of the Love type.

Knowledge of the phase velocity and attenuation dispersion curves of Love waves resulting from the theoretical model for Love waves in viscoelastic media can constitute the basis for an inverse (problem) method for determining the unknown physical and geophysical parameters of the actual structures of the Earth interior.

The results of this paper are original and fundamental contribution to the state-of-the-art. Since this paper covers properties of both high and low frequency Love waves the results obtained can be used in many fields of science and technology, such as: seismology and geophysics, non-destructive testing of materials, polymer research, and in the design and development of viscosity sensors, as well as bio and chemosensors.

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