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Inverse Method for Determining Profiles of Elastic Parameters in the Functionally Graded Materials using Love Waves

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Summary
This paper presents the use of SH (Shear Horizontal) surface Love waves to determine the distributions of elastic parameters in nonhomogeneous Functionally Graded Materials. The advantage of Love waves applied to investigate the elastic properties of materials is that the Love wave energy (in contrast to the other types of waves, e.g., plate Lamb waves) is concentrated in the vicinity of the surface layer. The penetration depth of the SH surface Love waves depends on the frequency. Therefore, Love waves are particularly suitable for investigating the profiles of the mechanical properties in nonhomogeneous Graded Materials. Direct Problem (Direct Sturm-Liouville Problem) that describes the propagation of Love waves in nonhomogeneous graded materials has been formulated and solved numerically by applying the Transfer Matrix Method. The Inverse Procedure (Inverse Sturm-Liouville Problem) for determining the distribution of elastic properties versus depth in the nonhomogeneous materials has been developed. Love wave dispersion curves in nonhomogeneous graded materials were evaluated numerically (synthetic data). Using the evaluated dispersion curves of Love waves and a developed Inverse Procedure the distributions of elastic shear coefficient as a function of depth (distance from the surface of the material into the bulk) in a heterogeneous surface layer deposited on a homogeneous substrate have been evaluated. Power type profiles (i.e., root square, linear and quadratic) of the shear elastic coefficient in the surface layer were considered. The results of this study can be useful in the investigation of elastic properties of Graded Materials in electronics as well as in geophysics and seismology.

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1. Introduction

The development of technology has led to the emergence of new and lighter materials with higher strength and more resistant to external factors. One of the new types of materials recently introduced into industrial practice are Functionally Graded Materials (FGM), [1, 2]. The mechanical properties of these materials vary in space. Usually these are nonhomogeneous materials in which mechanical parameters are functions of depth (distance from the surface into the bulk of the material).

The main properties determining the usefulness of bulk materials and thin films for industrial applications as well as their performance characteristics are the elastic properties [3].

Evaluation of mechanical parameters of FGM is of great practical importance in industrial applications, e.g., in electronics (to optimize the performance of electron devices and MEMS – Micro Electromechanical Systems) and in optoelectronics (to design an optimal construction of semiconductor lasers).

For the measurement of geometrical and mechanical parameters of surface layers and bulk materials classical methods mostly mechanical methods have been used [4], e.g., 1) surface profilometry, 2) indentation, 3) metallographic section, 4) measurement of residual stresses, 5) X-ray diffraction, 6) neutron scattering. These methods are very cumbersome, time consuming and destructive (e.g., indentation method).

Recently, ultrasonic methods have been introduced to measure the physical properties of materials [5, 6, 7, 8, 9, 10, 11, 12, 13]. Measured ultrasonic parameters (velocity and attenuation of the wave) are strongly dependent on the microstructure and mechanical properties of materials. The development of electronics has enabled the accurate measurement of ultrasonic parameters, e.g., determination of wave velocity, from the measurement of elastic wave pulse trains time of flight.

In this work, to investigate elastic properties of nonhomogeneous Functionally Graded Materials the authors used ultrasonic Love waves. Love waves are shear horizontal (SH) surface waves [14]. Love waves have been

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used to characterize physical parameters of solids and liquids [15, 16, 17, 18, 19, 20, 21, 22] as well as in geophysics [23, 24].

The advantage of Love waves in relation to the surface Rayleigh waves is that they have only one component of the mechanical displacement, in contrast to Rayleigh waves, which have two components. For this reason, the mathematical description of the propagation of surface Love waves in the nonhomogeneous graded materials considerably simplifies. Love wave energy (in contrast to the other types of waves, e.g., plate Lamb waves) is concentrated in the vicinity of the surface layer. The penetration depth of the SH surface Love waves depends on the frequency. Therefore, Love waves are particularly suitable for investigating the profiles of the mechanical properties in nonhomogeneous Functionally Graded Materials.

The aim of the studies carried out by the authors was to determine profiles of shear elastic modulus changes in Graded Materials on the example of a nonhomogeneous elastic surface layer deposited on a homogeneous substrate. In considered Graded Materials, shear modulus of elasticity is a continuous function of depth (distance from the surface). In this purpose, the authors formulated the Inverse Method for determining profiles of the shear elastic coefficient changes as a function of depth (distance from the treated surface area), based on knowledge of the dispersion curves of the shear Love wave, which propagates in the considered nonhomogeneous elastic waveguides.

In order to determine the Love wave dispersion curves numerical experiment was carried out. The exact dispersion curves (determined from the solution of the Direct Sturm-Liouville Problem) have been distorted by random noise with a preset level. Obtained in this manner dispersion curves (synthetic data) are treated as experimental Love wave dispersion curves. These synthetic dispersion curves were subsequently used in numerical calculations of the Inverse Method.

In this work, the Direct Sturm-Liouville Problem that describes the propagation of shear surface Love waves in the considered nonhomogeneous elastic graded material has been solved employing the Transfer Matrix Method. Using a developed Inverse Method, the authors determined the shear elastic coefficient profiles in the considered layered nonhomogeneous Graded Materials.

Formulation and solution of the Direct Problem and Inverse Problem for the Love wave propagating in the considered waveguide structures (see Figure 1), in which the elastic properties vary continuously with depth, is a novelty.

The results of this work can be applied in the investigation of Graded Materials applied in the aviation, aerospace and electronic industry, in fine mechanics, biomaterials as well as in geophysics and seismology.

2. Direct Sturm-Liouville Problem

Consider the propagation of Love waves in inhomogeneous elastic half-space in which the shear elastic coefficient is a continuous function of depth, see Figure 1. Such structures may represent elastic media occurring in the structures used, among others, in the electronics, aerospace and astronautic industry.

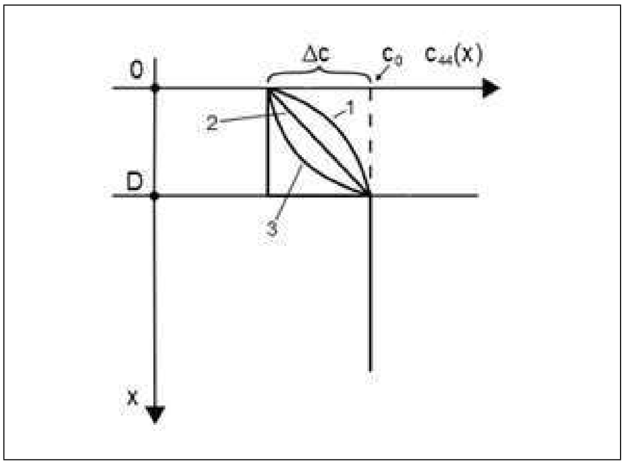

Figure 1. Variation of the elastic coefficient $c_{44}(x)$, as a function of depth, in a non-homogeneous elastic graded layer deposited on a homogeneous elastic substrate.

Profiles of changes in the elastic coefficient $c_{44}(x)$ in a nonhomogeneous elastic layer deposited on a homogeneous surface (see Figure 1) are represented by the following formulas:

a) square root type profile $n = 1/2$ (profile no.1 in Figure 1)

$$c_{44}(x)/c_0 = 1 - \left(\Delta c/c_0\right)\left[1 - \left(x/D\right)^{1/2}\right] \cdot \left[H(x-D) - H(x)\right], \quad (1a)$$

b) linear profile (profile no.2 in Figure 1)

$$c_{44}(x)/c_0 = 1 - \left(\Delta c/c_0\right)\left[1 - x/D\right] \cdot \left[H(x-D) - H(x)\right], \quad (1b)$$

c) quadratic profile (profile no.3 in Figure 1)

$$c_{44}(x)/c_0 = 1 - \left(\Delta c/c_0\right)\left[1 - \left(x/D\right)^{2}\right] \cdot \left[H(x-D) - H(x)\right], \quad (1c)$$

where $H(x)$ is the Heaviside step function, D is the depth of an inhomogeneous elastic layer.

Love wave is a Shear Horizontal (SH) surface wave which has only one component of the mechanical displacement (along the y axis) perpendicular to the direction of propagation (z-axis) and parallel to the surface ($z = 0$) of the waveguide. Love wave that propagates in an inhomogeneous elastic waveguides is presented in Figure 2.

Shear horizontal surface Love waves propagating in the considered heterogeneous elastic waveguide can be represented in the form

$$u = f(x) \exp\left(\mathrm{j}(\beta z - \omega t)\right),$$

where $f(x)$ is the amplitude of the Love wave, β is a propagation constant of the wave, $\mathrm{j} = \sqrt{-1}$, x is the distance

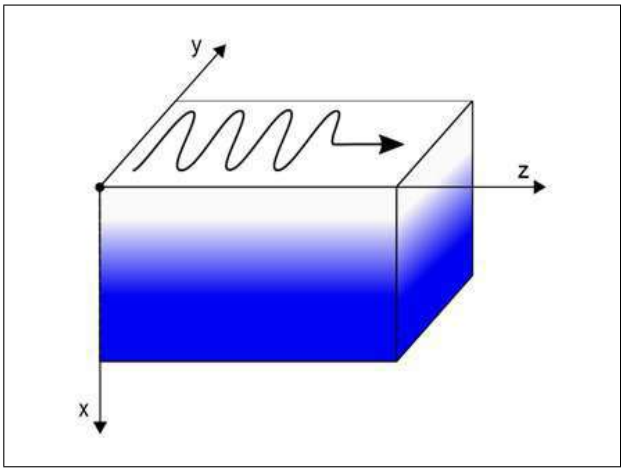

Figure 2. Love waves propagate in the direction of the z-axis. Mechanical vibration of the Love wave is performed along the y-axis. The shear elastic modulus $c_{44}(x)$ of the inhomogeneous elastic waveguide varies continuously with the depth x.

from the surface (depth), z is the direction of wave propagation and ω is an angular frequency. Love wave mechanical movement is performed along the y-axis.

Mechanical field generated by Love waves propagating in an inhomogeneous elastic graded medium satisfies the boundary conditions
a) on a free surface ($x = 0$), the transverse shear stress

$$\tau_{yx} = c_{44}(0)\frac{\mathrm{d}f(0)}{\mathrm{d}x}\exp\left(\mathrm{j}(\beta z - \omega t)\right)$$

is equal to zero, hence $\mathrm{d}f(0)/\mathrm{d}x = 0$,
b) at large distances ($x \to \infty$) from the surface ($x = 0$) the mechanical displacement of the Love wave should tend to zero, i.e., $f(\infty) = 0$.

The equation of motion for Love waves propagating in an inhomogeneous elastic medium (isotropic and in certain specified directions in media with regular and hexagonal symmetry) is represented by the Differential Problem [25]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(c_{44}(x)\frac{\mathrm{d}f}{\mathrm{d}x}\right) + \rho\omega^2 f = c_{44}(x)\beta^2 f, \quad (2)$$

$$\frac{\mathrm{d}f(0)}{\mathrm{d}x} = 0, \qquad f(\infty) = 0. \quad (3)$$

The Differential Problem (2 and 3) is named the Direct Sturm-Liouville Problem. The solution of the Direct Sturm-Liouville Problem is a set of pairs $(\beta_i^2, f_i(x))$, where β_i^2 is the ith eigenvalue, $i = 1, 2, \ldots, n$, n is the number of modes of Love waves propagating in considered waveguide and $f_i(x)$ is the eigenvector corresponding to this eigenvalue. Eigenvalue corresponds to the phase velocity of the propagating surface Love wave, while the eigenvector describes the distribution of the mechanical displacement of an appropriate mode of the surface wave as a function of depth. The set of phase velocities of the surface Love wave for various values of frequency determines the Love wave dispersion curve.

In the present paper, we restricted our analysis of the propagation of Love waves in graded materials to the fundamental ($i = 1$) mode of Love waves. The reason that the authors analyzed only the fundamental mode of Love waves was as folllows:
A) The energy contained in the higher modes is lower than the energy carried by the fundamental mode.
B) Mechanical displacement of the fundamental mode of Love waves reaches a maximum value on the surface of the waveguide. By contrast, the mechanical displacement of the higher modes of Loves wave attains a maximum value at some distance from the surface, in the bulk of the material.
C) The penetration depth for higher modes of Love waves is greater than the penetration depth for the fundamental mode.
D) The concentration of the Love wave energy in the vicinity of the surface for the fundamental mode is large. The concentration of energy in the subsurface region for higher modes is much smaller (due to their large penetration depth). For this reason, sensors of physical quantities of liquids (e.g., viscosity and density) based on the use of the fundamental mode of Love waves have the highest sensitivity.

All the above mentioned properties of the Love waves motivate the use only fundamental mode in the Love wave based bio and chemo-sensors. The constant density of the considered graded materials $\rho = \rho_0$ =const. was assumed throughout the paper.

2.1. Solution of the Direct Sturm-Liouville Problem

Solution of the Sturm-Liouville Direct Problem (2-3) for arbitrary function $c_{44}(x)$ is possible only numerically. Therefore, in the case of power type profiles (square root, linear and quadratic) of the elastic stiffness $c_{44}(x)$, the Direct Sturm-Liouville Problem was solved numerically. To this end, we applied the Transfer Matrix Method, that is a numerical algorithm developed to analyze seismic wave propagation in nonhomogeneous media in geophysics.

In the Transfer Matrix Method the nonhomogeneous elastic waveguide is divided (along the vertical x-axis) into a finite number of homogeneous layers [26, 27]. In each homogeneous layer, an ordinary differential equation of second order (Equation 2) occurring in the differential Sturm-Liouville Problem is replaced by the system of two ordinary differential equations of the first order. Here, the mechanical displacement of the Love wave and shear stress are the unknowns. Solving this set of differential equations in a layer, from knowledge of the mechanical displacement and shear stress on the upper surface of the layer, we can determine the mechanical displacement and shear stress on the lower surface of the layer. By performing this operation for each layer we can link the mechanical displacement and shear stress on the upper surface of the domain with the mechanical displacement and shear stress on the lower surface of the domain.

Imposing the appropriate boundary conditions on this two boundary surfaces leads to the dispersion equation for the Love wave. This equation is nonlinear algebraic equation for the unknown β^2, where β is the wave number of the Love wave. Thus, the phase velocity of the Love wave amounts to $v_p = \omega/\beta$. Set of pairs (v_p, ω) determines the phase velocity dispersion curves of the Love wave.

3. Inverse Sturm-Liouville Problem

Generally speaking, the Inverse Problem consists in determining the input data (e.g., elastic coefficients of the medium) based on knowledge of the effects of the considered physical phenomenon e.g., measured dispersion curves of elastic wave propagating in the investigated material [28]. The basis for solving Inverse Problems is the ability to solve efficiently the Direct Problem.

Inverse Problem considered in this work relies on the determination of unknown elastic parameters from the measured dispersion curves (phase velocity as a function of frequency) for Love waves that propagate in an inhomogeneous elastic waveguide. To solve the Inverse Problem one has to perform the following steps:

- solve Direct Problem,
- determine experimentally dispersion curves,
- solve Inverse Problem.

In this paper, the Inverse Problem was formulated and solved as an optimization problem [28] with properly defined objective function.

3.1. Objective function

The objective function is a measure of the distance between the mathematical model of the investigated object and the real object. The objective function depends on the distribution of the elastic coefficient $c_{44}(x)$ in nonhomogeneous investigated elastic structure, frequency, and experimental data (phase velocity of the surface Love wave). The nonhomogeneous elastic layer from Figure 1 was divided into 10 homogeneous layers. Values of the $c_{44}(x)$ coefficient at 9 evenly spaced discrete points of the surface layer, i.e., $c_{44}(x_i)$, $i = 1, 2, \ldots, 9$ are treated as the unknowns to be determined from the Inverse Procedure. The objective function was introduced and defined as

$$\Pi(t_1, t_2, \ldots, t_9) = \sum_{i=1}^{N_e} \left\{ \left(\frac{v_i^{\exp} - v_i^{\mathrm{cal}}(t_1, t_2, \ldots, t_9)}{v_i^{\exp}} \right)^2 \right\}, \tag{4}$$

where N_e is the number of experimental frequencies, $v_i^{\exp}$ is the measured phase velocity, v_i^{cal} is the calculated phase velocity, $t_1 = c_{44}(x_1)$, $t_2 = c_{44}(x_2)$, …, $t_9 = c_{44}(x_9)$, represent operational variables, that are determined from the solution of the Inverse Sturm-Liouville Problem.

The objective function Π represents the distance between the data obtained from the experiment $v_i^{\exp}$ and data predicted from the theoretical model v_i^{cal}. Optimized are elastic parameters of the layer, i.e., $c_{44}(x)$, whereas the material parameters of the substrate are given ($c_{44}(x_{10}) = c_0$ and ρ_0).

Making use of the optimization methods a minimum of the objective function was determined. This enabled the determination of the optimum values for the unknown distribution of the elastic coefficient $c_{44}(x)$ in the nonhomogeneous graded layer. To minimize the considered objective function Π the appropriate optimization procedures from the Scilab software package were employed [29, 30].

To solve the nonlinear minimization problem the optimization numerical procedures of the Nelder-Mead type [31] were used. In this solving routine, guess initial values for the unknowns are required. Initial guess values should be taken from the physical conditions. Their values must be realistic.

3.2. Numerical experiment (synthetic data)

In this study, the following values of material parameters of the considered nonhomogeneous elastic layer of depth D from Figure 1 were used:

$$c_0 = 2.564 \cdot 10^{10}\,\mathrm{N/m^2}, \qquad v_0 = 1849\,\mathrm{m/s},$$
$$\rho_0 = 7.5 \cdot 10^3\,\mathrm{kg/m^3}, \qquad \Delta c/c_0 = 0.088.$$

v_0 is the velocity of the bulk shear wave in the substrate. Homogeneous elastic half-space extends from $x = D$ to $x = \infty$.

These parameters are typical for PZT-4 [32] ceramics with elastic properties perturbed in the vicinity of the surface.

In our numerical experiments we determined the dispersion curves numerically. In the first step, by solving the Direct Problem (Equations 2 and 3), we calculated the phase velocity curves (see Figures 3, 4 and 5) for the exact values of the elastic coefficient from Figure 1. The dispersion curves represented in Figures 3, 4 and 5 by solid lines (not corrupted by noise) are regarded as exact dispersion curves.

Subsequently, we added random errors (e.g., 1%, 5%, 10%) to these values of the phase velocity. Obtained in this way dispersion curves are treated in the Inverse Procedure as simulated experimental curves. Points marked in Figures 3, 4 and 5 by diamond symbols indicate exemplary synthetic (experimental) dispersion curves obtained by corrupting the exact dispersion curves by random noise on the level of $\pm 1\%$.

The upper and lower dashed lines delimit the synthetic dispersion curves of phase velocity which results from exact dispersion curves (solid lines) that are subject to corruption by $\pm 1\%$ maximum random noise. Dispersion curves were evaluated and disturbed by random noise for 6 various values of normalized frequency D/L: $D/L =$ 0.5, 1.0, 2.0, 3.0, 4.0 and 5.0.

Numerical calculations have been performed by using the software package Scilab. In the numerical calculations, we assumed the value of thickness $D/L = 0.4$ mm. In this case, the value of $D/L = 1$ corresponds to frequency 4.6225 MHz, and $D/L = 5$ corresponds to frequency 23.11 MHz. Measurement of surface waves velocity in the MHz frequency range are usually performed in quantitative nondestructive evaluation experiments (QNDE).

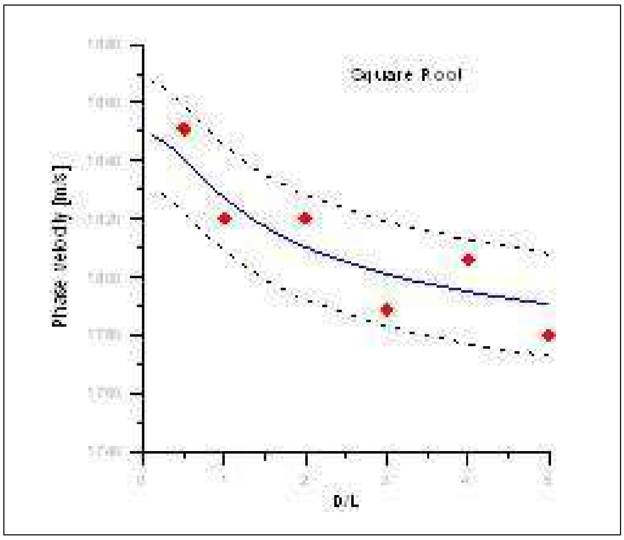

Figure 3. Phase velocity dispersion curves of Love wave propagating in a nonhomogeneous Graded elastic surface layer deposited on a homogeneous substrate. Elastic coefficient $c_{44}(x)$ in the surface layer varies according to the square root function of the depth.

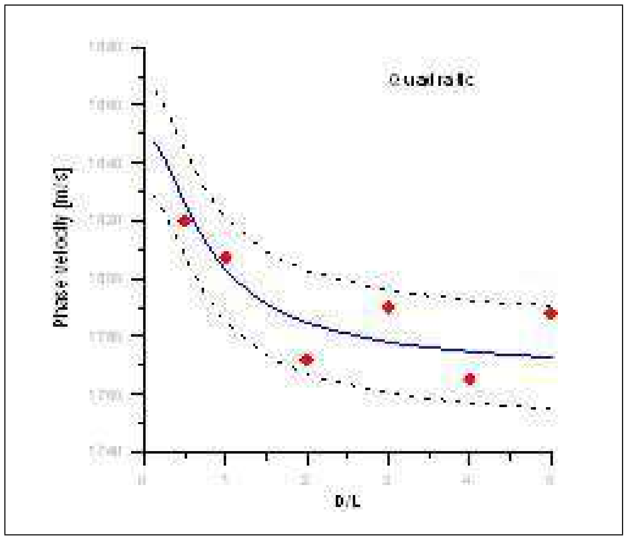

Figure 5. Phase velocity dispersion curves of Love wave propagating in a nonhomogeneous Graded elastic surface layer deposited on a homogeneous substrate. Elastic coefficient $c_{44}(x)$ in the surface layer follows the quadratic function of the depth.

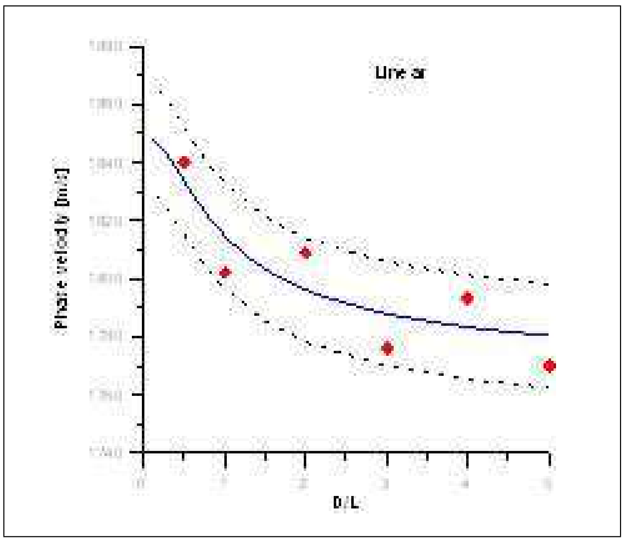

Figure 4. Phase velocity dispersion curves of Love wave propagating in a nonhomogeneous Graded elastic surface layer deposited on a homogeneous substrate. Elastic coefficient $c_{44}(x)$ in the surface layer is a linear function of the depth.

4. Results of numerical calculations and discussion

The Direct Problem that describes the propagation of Love waves in nonhomogeneous elastic Graded Materials was formulated and solved numerically by employing the Transfer Matrix Method [26, 27]. Theoretical (exact) dispersion curves for the Love surface waves, propagating in the selected nonhomogeneous structures, were solutions of the Direct Problem.

The Inverse Problem was formulated and solved as an Optimization Problem. Consequently, the objective function depending on the unknown distribution of the elastic coefficient $c_{44}(x)$ in nonhomogeneous Graded Materials was determined and minimized. The minimization problem was solved using the computer program implemented in Scilab software package. The Nelder-Mead algorithm was employed, that uses the concept of Simplex along with nonlinear optimization [29, 30].

The nonhomogeneous surface layer from Figure 1, $x \in [0, D]$ was divided into 10 homogeneous elastic layers. Unknown values of the elastic coefficient $c_{44}(x)$ are determined in 9 evenly spaced points $[x_1, x_2, \ldots, x_9]$ at the layers' boundaries. Thus, an unknown vector of the elastic coefficient is sought in the form of

$$c_{44}^{\text{eval}} = \left[c_{44}(x_1), c_{44}(x_2), \ldots, c_{44}(x_9)\right]^{\text{T}}.$$

Minimization of the objective function subject to the given constraints ($c_0 - \Delta c < c_{44}(x_i) < c_0$, $i \in [1 \ldots 9]$ results in the optimum values of unknown parameters (i.e., the distribution of shear elastic coefficient c_{44}^{eval} with depth).

Figure 6 illustrates an exemplary distribution of elastic coefficient c_{44}^{eval} in the surface layer as a function of depth for the square root type profile obtained using the Inverse Method (dotted line). Numerical experiment has been conducted for random noise level of 1%. Solid line represents an exact profile given by plot no.1 in Figure 1.

Exemplary distribution of changes in the elastic coefficient c_{44}^{eval} in the surface layer for the linear type profile, resulting from the application of the Inverse Method, is presented in Figure 7. Numerical experiment has been performed for random noise level of 1%.

Figure 8 shows (obtained from the Inverse Method) an exemplary distribution of the elastic coefficient c_{44}^{eval} in the surface layer as a function of depth for a quadratic type profile. Numerical experiment has been performed for random noise level of 1%.

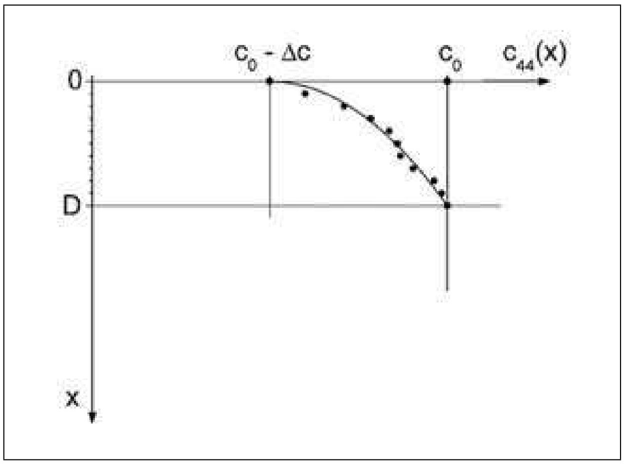

Figure 6. Elastic coefficient c_{44}^{eval} evaluated from the Inverse Problem (dotted line). Solid line represents an exact distribution of the shear elastic coefficient $c_{44}(x)$ for square root type profile (given by Equation 1a).

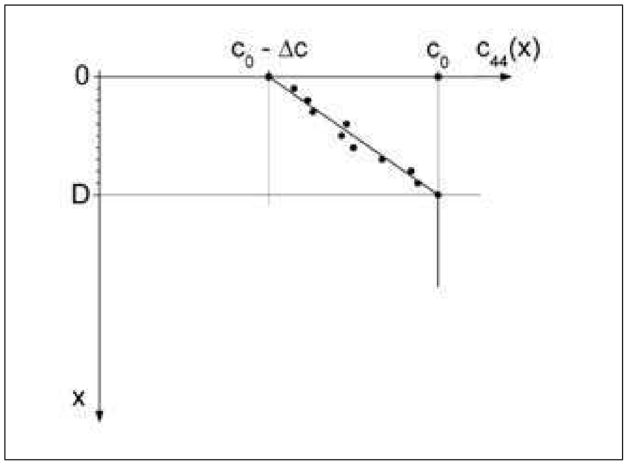

Figure 7. Elastic coefficient c_{44}^{eval} evaluated from the Inverse Problem (dotted line). Solid line represents an exact distribution of the shear elastic coefficient $c_{44}(x)$ for the linear type profile (given by Equation 1b).

4.1. Relative error

The relative error is a quantitative measure of the distance between the evaluated (from the Inverse Method) profile of the elastic coefficient c_{44}^{eval} and the exact profile c_{44}^{exact} of the elastic coefficient from Figure 1, e.g., linear, quadratic etc.

Using the concept of norm $||\cdot||$ (introduced by the Polish mathematician Stefan Banach), relative error of a single measurement of the elastic coefficient $c_{44}(x)$ can be defined as follows: Relative Error $= ||c_{44}^{\text{eval}} - c_{44}^{\text{exact}}||/||c_{44}^{\text{exact}}||$.

In this work, as the norm of the numerical sequence, the l_1 norm was chosen. This norm is the sum of modulus of subsequent sequence elements. In this way, the relative error ($R\,err$) of a single measurement (evaluation) of the elastic coefficient $c_{44}(x)$ amounts to

$$(R\,err)_{N=1} = \frac{||c_{44}^{\text{eval}} - c_{44}^{\text{exact}}||_{l_1}}{||c_{44}^{\text{exact}}||_{l_1}} \tag{5}$$
$$= \Big[|c_{44}^{\text{eval}}(x_1) - c_{44}^{\text{exact}}(x_1)| + |c_{44}^{\text{eval}}(x_2) - c_{44}^{\text{exact}}(x_2)| + \cdots + |c_{44}^{\text{eval}}(x_9) - c_{44}^{\text{exact}}(x_9)|\Big]$$
$$\Big[|c_{44}^{\text{exact}}(x_1)| + |c_{44}^{\text{exact}}(x_2)| + \cdots + |c_{44}^{\text{exact}}(x_1)|\Big]^{-1}.$$

Similarly, the relative error for a series of N evaluations (from the solution of the Inverse Problem) of the distribution of the elastic coefficient $c_{44}(x)$, is defined as

$$(R\,err)_N = \left\{ \frac{||(c_{44}^{\text{eval}})_1 - c_{44}^{\text{exact}}||_{l_1}}{||c_{44}^{\text{exact}}||_{l_1}} + \frac{||(c_{44}^{\text{eval}})_2 - c_{44}^{\text{exact}}||_{l_1}}{||c_{44}^{\text{exact}}||_{l_1}} + \cdots + \frac{||(c_{44}^{\text{eval}})_N - c_{44}^{\text{exact}}||_{l_1}}{||c_{44}^{\text{exact}}||_{l_1}} \right\} \Big/ N. \tag{6}$$

For subsequent profiles of the elastic modulus from Figure 1, a series of $N = 10$ numerical measurements of Love wave dispersion curves was conducted. To this end, using a random number generator, for each profile 10 different

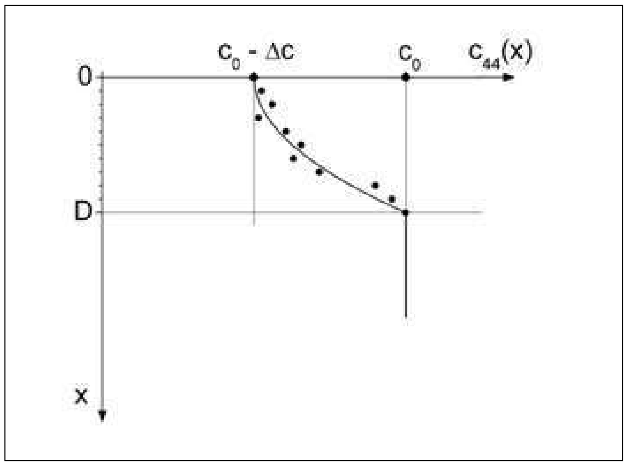

Figure 8. Elastic coefficient c_{44}^{eval} evaluated from the Inverse Problem (dotted line). Solid line represents an exact distribution of the shear elastic coefficient for quadratic type profile (given by Equation 1c).

dispersion curves of Love waves were evaluated corrupting the exact dispersion curve by the random error of a specific level. Each of these dispersion curves (synthetic data), was used in the calculations of the Inverse Method. Using, obtained in such a manner, elastic coefficient profiles c_{44}^{eval}, the relative error of determining the distribution of the elastic coefficient $c_{44}(x)$, treated as a function of depth, has been determined, see Table I.

Table I contains the relative error for the series of $N = 10$ evaluations of the unknown distribution of the modulus of elasticity $c_{44}(x)$ for the subsequent profiles from Figure 1 and the different levels of random noise. 10 inversions for each type of the profile and each level of noise have been performed. In Table I, the average values of 10 subsequent inversions are inserted.

As can be seen from Table I and Figures 6, 7 and 8, the proposed Inverse Method can be effectively used to identify the modulus of elasticity $c_{44}(x)$ profile changes in

Table I. Relative error $(\text{Relativeerror})_{N=10}$ of the determination of the elastic coefficient $c_{44}(x)$ evaluated from the Inverse Method, for the maximum random errors equal to 0.1%, 1%, 5%, and 10%. Each evaluation of the elastic coefficient (c_{44}^{eval}) results from the minimization of the objective function Π (Equation 4), for subsequent simulated dispersion curves of phase velocity.

Random error	0.1%	1%	5%	10%
square root profile [%]	3.59	9.93	13.31	16.21
linear profile [%]	4.58	9.39	13.52	15.42
quadratic profile [%]	2.68	6.31	9.98	14.67

Graded Materials. The accuracy of the obtained (from Inverse Method) modulus of elasticity $c_{44}(x)$ profile changes is good.

5. Conclusions

In this work, Love waves which are horizontally polarized surface waves were employed to determine the distribution of the elastic coefficient profiles in Graded Materials.

An Inverse Method that uses Love waves for determining the distribution of the shear elastic coefficient $c_{44}(x)$ in elastic Functionally Graded Materials, from evaluated dispersion curves, has been developed.

The advantage of Love waves with respect to the Rayleigh wave is that the Love wave has only one component of the mechanical displacement, in contrast to Rayleigh waves which possess two components of the mechanical displacement. For this reason, the mathematical description of the propagation of SH Love waves in Graded Materials is significantly simplified.

In the paper, the Sturm-Liouville Direct Problem for the Love wave propagating in a nonhomogeneous elastic layer deposited on the homogeneous substrate was formulated and solved using the Transfer Matrix Method. Subsequently, the Inverse Problem for the ultrasonic Love wave propagating in the considered inhomogeneous waveguide structure was also formulated and solved. The Inverse Problem was formulated and solved as an optimization problem.

Formulation and solution of the Direct Problem and Inverse Problem for the Love wave propagating in the considered elastic graded structures is a novelty.

The results obtained in this study can be helpful in determining profiles of elastic coefficients changes in various Graded Materials. Materials of this type are produced during technological processes used in many industries such as: electronic, aviation, aerospace, automotive as well as in medicine and biomechanics. Moreover, the results of this work can also be employed in geophysics and seismology for investigation structure and the elastic properties of the Earth's interior, [33].

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References

[1] S. Suresh, A. Mortensen: Fundamentals of functionally graded materials. The Institute of Materials, London, 1998.

[2] S. K. Bohidar, R. Sharma, P. R. Mishra: Functionally graded materials: a critical review. International Journal of Research **1** (2014) 289–301.

[3] J. Chang, Z. Yang, J. Xu: Inverse method for the determination of elastic properties of coating layers by the surface ultrasonic waves. Journal of Zhejiang University, Science **6** (2005) 945–949.

[4] R. Górecka, Z. Polański: Metrologia Warstwy Wierzchniej (in Polish). WNT, Warsaw, 1983.

[5] J. Krautkramer, H. Krautkramer: Ultrasonic testing of materials. Springer, New York, 1983.

[6] P. Kiełczyński, W. Pajewski, M. Szalewski: Piezoelectric sensors for investigations of microstructures. Sensors and Actuators A-Physical **65** (1998) 13–18.

[7] T. T. Wu: Elastic wave propagation and nondestructive evaluation of materials. Proceedings of the National Science Council, ROC(A) **23** (1999) 703–715.

[8] P. Kiełczyński, W. Pajewski, M. Szalewski: Admittance of axially polarized lossy piezoceramic cylinders loaded with a viscoelastic liquid. Journal of Applied Physics **91** (2002) 10181–10187.

[9] P. Kiełczyński, W. Pajewski, M. Szalewski: Determination of the shear impedance of viscoelastic liquids using cylindrical piezoceramic resonators. IEEE Trans on Ultrasonics, Ferroelectrics and Frequency Control **50** (2003) 230–236.

[10] P. Kiełczyński, M. Szalewski, R. M. Siegoczyński, A. J. Rostocki: New ultrasonic Bleustein-Gulyaev wave method for measuring the viscosity of liquids at high pressure. Review of Scientific Instruments **79** (2008) 026109–1–3.

[11] H. K. Oh, W. Wang, K. Lee, C. Min, S. Yang: The development of a wireless Love wave biosensor on 41grad YX $LiNbO_3$. Smart Materials and Structures **18** (2009) 025008.

[12] A. J. Rostocki, R. M. Siegoczynski, P. Kiełczyński, M. Szalewski, A. Balcerzak, M. Zduniak: Employment of a novel ultrasonic method to investigate high pressure phase transitions in oleic acid. High Pressure Research **31** (2011) 334–338.

[13] M. B. Amor, I. B. Salah, M. H. B. Ghozlen: Propagation behavior of Lamb waves on functionally graded piezoelectric plates. Acta Acustica united with Acustica **101** (2015) 435–442.

[14] J. D. Achenbach: Wave propagation in elastic solids. North-Holland, Amsterdam, 1973.

[15] P. Kiełczyński, W. Pajewski: Inverse method for determining the depth of nonhomogeneous surface layers in elastic solids from the measurements of the dispersion curves of group velocity of surface SH waves. Applied Physics A **48** (1989) 423–429.

[16] P. Kiełczyński, M. Szalewski: An inverse method for determining the elastic properties of thin layers using Love surface waves. Inverse Problems in Science and Engineering **19** (2011) 31–43.

[17] P. Kiełczyński, M. Szalewski, A. Balcerzak: Effect of a viscous liquid loading on Love wave propagation. International Journal of Solids and Structures **49** (2012) 2314–2319.

[18] P. Kiełczyński, M. Szalewski, A. Balcerzak, A. Malanowski, R. M. Siegoczyński, S. Ptasznik: Investigation of high-pressure phase transitions in DAG (diacylglycerol) oil using the Bleustein-Gulyaev ultrasonic wave method. Food Research International **49** (2012) 60–64.

[19] H. Dong, G. Ke, K. Duffaut: Seabed shear-wave velocity estimation from dispersion curves of Love waves. Acta Acustica united with Acustica **99** (2013) 980–897.

[20] P. Kiełczyński, M. Szalewski, A. Balcerzak: Inverse procedure for simultaneous evaluation of viscosity and density of Newtonian liquids from dispersion curves of Love waves. Journal of Applied Physics **116** (2014) 044902.

[21] I. B. Salah, M. B. Amor, M. H. B. Ghozlen: Effect of a functionally graded soft middle layer on Love waves propagating in layered piezoelectric systems. Ultrasonics **61** (2015) 145–150.

[22] A. Talbi, A. Soltani, A. Rumeau, A. Taylor, L. Drbohlavova, L. Klimsa, J. Kopecek, L. Fekete, M. Krecmarova, V. Mortet: Simulations, fabrication, and characterization of diamond-coated Love wave-type surface acoustic wave sensors. Physica Status Solidi A **1-5** (2015) DOI 10.1002/pssa.201532188.

[23] T. Boxberger, M. Picozzi, S. Parolai: Shallow geology characterization using Rayleigh and Love wave dispersion curves derived from seismic noise array measurements. Journal of Applied Geophysics **75** (2011) 345–354.

[24] M. W. Morrison: In-situ viscoelastic soil parameter estimation using Love wave inversion. Ph. D. Dissertation, Boise State University, Department of Geoscience, USA, 2014.

[25] P. Kiełczyński, M. Szalewski, A. Balcerzak, K. Wieja: Group and phase velocity of Love waves propagating in elastic functionally graded materials. Archives of Acoustics **40** (2015) 273–281.

[26] N. A. Haskell: The dispersion of surface waves on multilayered media. Bulletin of the Seismological Society of America **43** (1953) 17–34.

[27] W. T. Thompson: Transmission of elastic waves through a stratified solid medium. Journal of Applied Physics **21** (1950) 89–93.

[28] G. R. Liu, X. Han: Computational inverse techniques in nondestructive evaluation. CRC Press, London, 2003.

[29] C. Gomez (ed.): Engineering and scientific computing with Scilab. Springer-Science+Bussiness Media, LLC, New York, 1999.

[30] G. E. Uroz: Numerical and statistical methods with SCILAB for science and engineering. vol. II, ch. 12, www.greatunpublished.com, Title No. 305, Charleston, S.C., USA, 2001.

[31] A. R. Conn, K. Scheinberg, L. N. Vincente: Introduction to derivative-free optimization. SIAM, Philadelphia, 2009.

[32] D. A. Berlincourt, D. R. Curran, H. Jaffe: Piezoelectric and piezomagnetic materials and their function in transducers. – In: Physical Acoustics, Vol. I, Part A. W. P. Mason (ed.). Academic Press, New York, 1964, Ch.3.

[33] S. Foti, C. G. Lai, G. J. Rix, C. Strobbia: Surface wave methods for near-surface site characterization. CRC Press, London, 2015.